



Construction of a mathematical Model of early and late Potato Blight based on the stochastic SICR Model

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Abstract

As a devastating fungal disease, that seriously threatens potato production, the transmission process of potato early and late blight is significantly affected by environmental fluctuations. To accurately reveal the epidemic regularity of this disease, a stochastic SICR (Susceptible-Infected-Confined-Recovered) model is a random, epidemiological framework used to analyze disease transmission model for potato early and late blight was constructed in this paper. Through dynamical analysis, first, it was proved that the system has a unique global positive solution, which ensures the biological rationality of the model; second, based on Has'minskii's theory, the sufficient conditions for the existence of the ergodic stationary distribution of the model were derived, clarifying the dynamical mechanism underlying the persistent epidemic of the disease; third, the threshold conditions for disease extinction and persistence in mean were established, identifying the key nodes for disease prevention and control. The results show that when $R_0^s > 1$, the model has an ergodic stationary distribution and the disease persists in epidemic; when $\overline{R_0} < 1$, the disease will eventually become extinct. This study provides important theoretical support for the risk early warning and precise prevention and control of potato early and late blight.

Keywords: Potato early and late blight, Stochastic SICR model, Stationary distribution, Disease Extinction, Disease Persistence.

1. INTRODUCTION

This document was prepared using the AIP Conference Proceedings template for Microsoft Word. It provides a simple example of a paper and offers guidelines for preparing your article. Here we introduce the paragraph styles for Level 1, Level 2, and Level 3 headings. Please note the following: Potato, as the fourth largest food crop in the world, plays a pivotal role in safeguarding food security and boosting agricultural economic development [1]. Potato early and late blight, caused by *Alternaria solani* and *Phytophthora infestans* respectively, are the most widely distributed and devastating fungal diseases in potato production [2-3]. Characterized by rapid transmission, wide epidemic range and heavy yield loss, these diseases can lead to a 20%–50% reduction in potato output and even total crop failure in severe cases [4-6]. The transmission process of potato early and late blight is complex, which is not only affected by biological factors such as pathogenic infection characteristics and host resistance, but also disturbed by the random fluctuations of environmental factors including temperature, humidity and precipitation [7-9]. Thus, traditional deterministic models are difficult to accurately depict their actual transmission dynamics.

Mathematical models are important tools for studying the transmission dynamics of infectious diseases and plant diseases. In recent years, numerous scholars have carried out research on the transmission mechanisms of infectious diseases and plant diseases by constructing dynamic models. For instance, Song et al. proposed a viral infection dynamic

model with a saturated infection rate [10-12]. Based on the stage-specific characteristics of disease transmission, Qiao et al. first proposed the SICR epidemic model, which divides the population into susceptible, acutely infected, pathogen-carrying and recovered classes, providing a new modeling approach for the study of stage-specific disease transmission [13]. This model has been successfully applied to the transmission analysis of various diseases with latent infection or carrier states, but it has not yet been applied to the research on potato early and late blight.

The transmission process of potato early and late blight exhibits distinct stage-specific characteristics: healthy potato plants (susceptible class) first show acute disease symptoms (acutely infected class) after being exposed to pathogens; some infected plants may shift to a latent infection state and become pathogen carriers (pathogen-carrying class); through their own immune responses or artificial prevention and control measures, some infected plants can recover and regain health (recovered class). This transmission process is highly consistent with the structure of the SICR model. However, in real-world environments, random fluctuations of environmental factors such as temperature, humidity and precipitation directly affect the infection efficiency and transmission rate of pathogens [13-18]. The deterministic SICR model fails to take these random disturbances into account, resulting in deviations in prediction results. Therefore, introducing random perturbations to construct a stochastic transmission model of potato early and late blight is of great significance for improving the accuracy of disease epidemic prediction.

Inspired by the above studies, this paper constructs a stochastic SICR transmission model based on the deterministic SICR model, with the impact of environmental noise on the transmission of potato early and late blight taken into consideration. Through dynamical analysis, we reveal the dynamical behaviors of the disease such as extinction and persistent epidemic under environmental fluctuations, which provides a theoretical basis for the precise prevention and control of this disease.

The structure of this paper is organized as follows: Section 2 constructs the stochastic SICR model for potato early and late blight and proves the existence and uniqueness of its global positive solution; Section 3 analyzes the existence of the ergodic stationary distribution of the model; Section 4 derives the conditions for disease extinction and persistence in mean.

2. PROBLEM STATEMENT

2.1 Model Construction and Existence and Uniqueness of Global Positive Solution.

Model Assumptions and Construction

Combined with the transmission characteristics of potato early and late blight, the following assumptions are made:

Divide the potato population into four categories: susceptible plants ($S(t)$), acutely infected plants ($I(t)$), pathogen-carrying plants ($C(t)$), and recovered plants ($R(t)$), among which recovered plants have immunity;

μ represents the natural growth rate and mortality rate of the total population;

γ_1 represents the recovery (or detachment) transition rate of acutely infected plants;

γ_2 represents the transition rate from pathogen-carrying plants to recovered plants;

β represents the contact rate between susceptible plants and acutely infected plants or pathogen-carrying plants;

$\alpha \in [0, 1]$ represents the increased transmission rate of acutely infected plants compared with pathogen-carrying plants;

$q \in [0, 1]$ represents the proportion of acutely infected plants transitioning to pathogen-carrying plants.

Based on the above assumptions, the SICR model is established as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \beta S(t)I(t) - \alpha\beta S(t)C(t) - \mu S(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) + \alpha\beta S(t)C(t) - \gamma_1 I(t) - \mu I(t), \\ \frac{dC(t)}{dt} = q\gamma_1 I(t) - \gamma_2 C(t) - \mu C(t), \\ \frac{dR(t)}{dt} = (1 - q)\gamma_1 I(t) + \gamma_2 C(t) - \mu R(t), \end{cases} \quad (1)$$

Since the fourth equation of system (1) is decoupled (it does not affect the dynamics of the first three equations), we can simplify system (1) to the following subsystem:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \beta S(t)I(t) - \alpha\beta S(t)C(t) - \mu S(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) + \alpha\beta S(t)C(t) - \gamma_1 I(t) - \mu I(t), \\ \frac{dC(t)}{dt} = q\gamma_1 I(t) - \gamma_2 C(t) - \mu C(t), \end{cases} \quad (2)$$

The initial condition of system (2) is $(S(0), I(0), C(0)) \in R_+^3$.

According to the theory in [7], system (2) has the following properties:

$\Gamma = \{(S(t), I(t), C(t)) \in R_+^3 : S(t) + I(t) + C(t) \leq 1\}$ is positively invariant for (2).

The disease free equilibrium $E_0 = (S_0, I_0, C_0) = (1, 0, 0)$, which always exists, is globally asymptotically stable if

$$R_0 = \frac{\beta}{\gamma_1 + \mu} + \frac{q\gamma_1}{\gamma_1 + \mu} \frac{\alpha\beta}{\gamma_2 + \mu} \leq 1.$$

When $R_0 > 1$, system (2) has a unique positive equilibrium E^* and it is globally asymptotically stable. However, in the real world, it is inevitable for the biological-mathematic models to be affected by environmental noise [19-23]. Hence, in order to make a more accurate predict for the dynamics nature of the epidemics system, it is more reliable for us to use stochastic differential equations models. Owing to this reason, many authors choose to add stochastic perturbations into the deterministic models for the sake of better account for the influences of the environmental fluctuations. For instance, Zhang, Jiang, Hayat and Ahmad study the dynamical behavior of stochastic SVIR models in the literature [19]. Wang, Jiang, Alsaedi, Hayat investigate a stochastic HIV viral model with both logistic target cell growth and nonlinear immune response function and derive the conditions of the extinction of the disease. In addition, they also prove the existence of the ergodic stationary distribution [24]. With respect to the rest of literatures, one can also refer to [25-32].

Inspired by the aforementioned research work, this paper introduces stochastic perturbations into Model (2). Considering the uncertainty of environmental factors such as temperature, humidity, and precipitation in the field, white noise is used to simulate such environmental perturbations, and these stochastic perturbations are proportional to the variables in System (2). Thus, the stochastic version corresponding to System (2) has the following form:

$$\begin{cases} dS(t) = [\mu - \beta S(t)I(t) - \alpha\beta S(t)C(t) - \mu S(t)]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = [\beta S(t)I(t) + \alpha\beta S(t)C(t) - \gamma_1 I(t) - \mu I(t)]dt + \sigma_2 I(t)dB_2(t), \\ dC(t) = [q\gamma_1 I(t) - \gamma_2 C(t) - \mu C(t)]dt + \sigma_3 C(t)dB_3(t), \end{cases} \quad (3)$$

Where $B_i(t)$, $i=1,2,3$, are the white noises, representing the impact of temperature, humidity, and precipitation fluctuations on the fungal transmission rate. For example, late blight spreads extremely rapidly when humidity exceeds 90%, and such environmental fluctuations are reflected as random terms in the model. In other word, $B=(B_1(t), B_2(t), B_3(t), t \geq 0)$ is mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{F_t, t \geq 0\}$ satisfying the general conditions (i.e., it is increasing and right continuous while F_0 contains all P-null sets), $\sigma_1, \sigma_2, \sigma_3$ stands for the intensities of the white noise and $\sigma_i > 0$ ($i=1,2,3$). All the other parameters have the same meaning as that of the system (2).

2.2 Existence and uniqueness of the global positive solution

This section deduces that for any initial value, the solution of System (3) is globally existent and always positive. As we all know, to prove that a stochastic differential equation has a unique global solution (i.e., no explosion in finite time) for any initial value, it is usually required that the coefficients of the system satisfy the linear growth condition and the local Lipschitz condition [33]. However, though the coefficients of system (3) are locally Lipschitz continuous, they do not satisfy the linear growth condition, hence the solution of system (3) may explode at a finite time. In the following, we will prove that there is a unique global positive solution of system (3) for any initial value.

Theorem 1. For any initial value $(S(0), I(0), C(0)) \in R_+^3$, there is a unique positive solution $(S(t), I(t), C(t))$ on $t \geq 0$ for system (3), further, the solution will remain in R_+^3 with probability one, in brief, $(S(t), I(t), C(t)) \in R_+^3$ for all $t \geq 0$ almost surely (a.s).

Proof. Since the coefficients of system (3) satisfy locally Lipschitz condition, so system (3) for any initial value exists a unique local solution $(S(t), I(t), C(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [34]. Next for the sake of illustrating this solution is global, it is significant for us to verify that $\tau_e = \infty$ a.s. At this time, let $k_0 \geq 1$ be enough large such that $(S(t), I(t), C(t))$ all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$\tau_k = \inf \{t \in [0, \tau_e): S(t) \notin (\frac{1}{k}, k) \text{ or } I(t) \notin (\frac{1}{k}, k) \text{ or } C(t) \notin (\frac{1}{k}, k)\}$, where throughout this paper, we set $\inf \emptyset = \infty$ (as usual \emptyset is the empty set). Clearly τ_k is increasing when $k \rightarrow \infty$. Let $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show $\tau_\infty = \infty$, then $\tau_\infty = \infty$, and $(S(0), I(0), C(0)) \in R_+^3$ a.s. for all $t \geq 0$. That shows that we are requested to illustrate that $\tau_\infty = \infty$, a.s. in order to finish the proof. If this assertion is not true, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$.

Hence there is an integer $k \geq k_T$ such that for all $k \geq k_T$

$$P\{\tau_k \leq T\} > \varepsilon. \tag{4}$$

Define a C^2 — function $V: R_+^3 \rightarrow R_+$ by

$$V(S, I, C) = \left(S - b - b \ln \frac{S}{b} \right) + (I - 1 - \ln I) + fC.$$

Where b, f are positive constants to be determined later. The non negativity of this function can be attained from $u - 1 - \ln u \geq 0$ for any $u > 0$. Using Itô formula to V , we get:

$$\begin{aligned} dV &= \left(1 - \frac{b}{S}\right) dS + \left(1 - \frac{1}{I}\right) dI + f dC + \frac{1}{2} \frac{b}{S^2} (dS)^2 + \frac{1}{2} \frac{1}{I^2} (dI)^2 = \left(1 - \frac{b}{S}\right) [(\mu - \beta SI - \alpha \beta SC - \mu S) dt + \sigma_1 S dB_1(t)] \\ &+ \frac{1}{2} b \sigma_1^2 dt + \left(1 - \frac{1}{I}\right) [(\beta SI + \alpha \beta SC - \gamma_1 I - \mu I) dt + \sigma_2 I dB_2(t)] + \frac{1}{2} \sigma_2^2 dt + f [(q\gamma_1 I - \gamma_2 C - \mu C) dt + \sigma_3 C dB_3(t)] \\ &= LV dt + \sigma_1 (S - b) dB_1(t) + \sigma_2 (I - 1) dB_2(t) + f \sigma_3 C dB_3. \end{aligned} \tag{5}$$

where LV is defined by

$$\begin{aligned} LV &= \left(1 - \frac{b}{S}\right) (\mu - \beta SI - \alpha \beta SC - \mu S) + \frac{1}{2} b \sigma_1^2 + \left(1 - \frac{1}{I}\right) (\beta SI + \alpha \beta SC - \gamma_1 I - \mu I) + \frac{1}{2} \sigma_2^2 + f (q\gamma_1 I - \gamma_2 C - \mu C) \\ &= (b + 2)\mu - (\mu + \beta)S - \frac{b\mu}{S} - \frac{\alpha \beta SC}{I} + \gamma_1 + \frac{1}{2} b \sigma_1^2 + \frac{1}{2} \sigma_2^2 + (b\beta + fq\gamma_1 - \gamma_1 - \mu)I + (b\alpha\beta - f\gamma_2 - f\mu)C \\ &\leq (b + 2)\mu + \gamma_1 + \frac{1}{2} b \sigma_1^2 + \frac{1}{2} \sigma_2^2 + (b\beta + fq\gamma_1 - \gamma_1 - \mu)I + (b\alpha\beta - f\gamma_2 - f\mu)C. \end{aligned}$$

We choose $b = \frac{(\gamma_1 + \mu)(\gamma_2 + \mu)}{\beta(\gamma_2 + \mu + \alpha q\gamma_1)}$ and $f = \frac{\alpha(\gamma_1 + \mu)}{\gamma_2 + \mu + \alpha q\gamma_1}$ such that $(b\beta + fq\gamma_1 - \gamma_1 - \mu)I = 0$ and

$(b\alpha\beta - f\gamma_2 - f\mu)C = 0$, then we can attain

$$LV \leq (b + 2)\mu + \gamma_1 + \frac{1}{2} b \sigma_1^2 + \frac{1}{2} \sigma_2^2 := K. \tag{6}$$

In which K is a positive constant. So we have:

$$dV \leq K dt + \sigma_1 (S - b) dB_1(t) + \sigma_2 (I - 1) dB_2(t) + f \sigma_3 C dB_3(t) \tag{7}$$

Integrating of (7) from 0 to $\tau_k \wedge T$ and then taking the expectation on both sides, further according to Theorem 1.5.8 (ii) [36, 37], we can obtain:

$$EV(S(\tau_k \wedge T), I(\tau_k \wedge T), C(\tau_k \wedge T)) \leq V(S(0), I(0), C(0)) + KE(\tau_k \wedge T).$$

Therefore,

$$EV(S(\tau_k \wedge T), I(\tau_k \wedge T), C(\tau_k \wedge T)) \leq V(S(0), I(0), C(0)) + KT. \tag{8}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$ and by means of (4), we can obtain $P(\Omega_k) \geq \varepsilon$. Notice that for every $\omega \in \Omega_k$, there exists that $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ or $C(\tau_k, \omega)$

equals either k or $\frac{1}{k}$. Therefore, $V(S(\tau_k, \omega), I(\tau_k, \omega), C(\tau_k, \omega))$ is no less than either $k - 1 - \ln k$ or $\frac{1}{k} - 1 + \ln k$.

Consequently, we have

$$V(S(\tau_k, \omega), I(\tau_k, \omega), C(\tau_k, \omega)) \geq (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k\right) := H(k).$$

According to (8), we can attain:

$$\begin{aligned} V(S(0), I(0), C(0)) + KT &\geq E(I_{\Omega_k}(\omega)V(S(\tau_k \wedge T), I(\tau_k \wedge T), C(\tau_k \wedge T))) \\ &\geq P(\Omega_k(\omega))H(k) \\ &\geq \varepsilon H(k), \end{aligned}$$

where I_{Ω_k} stands for the indicator function of Ω_k . Letting $k \rightarrow \infty$, then

$$\infty > V(S(0), I(0), C(0)) + KT \geq \varepsilon H(k) \rightarrow \infty,$$

which lead to the contradiction. Thus, the proof is completed.

3. EXISTENCE OF ERGODIC STATIONARY DISTRIBUTION OF SYSTEM (3)

In the dynamic analysis of potato plants, we pay great attention to when *Alternaria solani* (early blight) and *Phytophthora infestans* (late blight) will break out and persistently prevail in farmland. The method of solving this problem for deterministic model is by means of the endemic equilibrium of the corresponding model, to be specific, we can prove that the endemic equilibrium of the system is globally asymptotically stable or is a global attractor. However, for the stochastic system (3), it does not have the endemic equilibrium. So in this section, we will derive the conditions of the existence of ergodic stationary distribution of system (3) according to the theory of Has'minskii [38]. In the following, we introduce some theories about stationary distributions, which is essential in the process of our proof [39-41].

Assuming that $X(t)$ is a regular time-homogeneous Markov process in the d —dimensional space E_d , and satisfy the following stochastic equation

$$dX(t) = b(X)dt + \sum_{r=1}^k g_r(X)dB_r(t).$$

At this time, the diffusion matrix has the following form

$$A(X) = (a_{ij}(x)), a_{ij}(x) = \sum_{r=1}^k g_r^i(x)g_r^j(x)$$

Lemma 1. (see [42, 43]) For Markov process $X(t)$, there is a unique ergodic stationary distribution $\pi(\cdot)$ if it exists a bounded domain $D \in E_d$ with regular boundary Γ and

(H1) there exists a constant $M > 0$ satisfying

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2, x \in D, \xi \in R^d.$$

(H2) there is a C^2 —function $V \geq 0$ such that LV is negative for any given $E_d \setminus D$. Then

$$P\{\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(X(t))dt = \int_{E_d} f(x)\pi(dx)\} = 1,$$

for all $x \in E_d$, where $f(\cdot)$ is an integrable function about the measure π .

Theorem 2. Assume that

$$R_0^S := \frac{\beta\mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)} + \frac{\alpha\beta q\gamma_1\mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)} > 1,$$

then system (3) has a unique stationary distribution $\pi(\cdot)$ and the ergodicity holds.

Proof. In order to prove Theorem 2, we need to verify the conditions of Lemma 1. First, we will verify that the condition (H2) in Lemma 1 is valid.

Define

$$V_1 = -\ln I - (a_1 + b_1)\ln S - b_2 \ln C + \frac{\alpha\beta(a_1 + b_1)}{\gamma_2 + \mu} C,$$

$$V_2 = -\ln S - \ln C + \frac{\alpha\beta}{\gamma_2 + \mu} C,$$

$$V_3 = \frac{1}{\theta + 1} (S + I + C)^{\theta + 1},$$

where a_1, b_1, b_2 are positive constants to be determined later. $\theta \in (0, 1)$ satisfy

$$\mu - \frac{\theta}{2} (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) > 0.$$

Applying the Itô formula, we can attain

$$\begin{aligned} LV_1 &= -\beta S - \frac{\alpha\beta SC}{I} + \gamma_1 + \mu + \frac{1}{2}\sigma_2^2 - \frac{a_1\mu}{S} - \frac{b_1\mu}{S} + \beta(a_1 + b_1)I + \frac{\alpha\beta q\gamma_1(a_1 + b_1)}{\gamma_2 + \mu} I \\ &\quad + \mu(a_1 + b_1) + \frac{1}{2}(a_1 + b_1)\sigma_1^2 - \frac{b_2 q\gamma_1 I}{C} + b_2\gamma_2 + b_2\mu + \frac{1}{2}b_2\sigma_3^2 \\ &= -\beta S - \frac{a_1\mu}{S} + a_1(\mu + \frac{1}{2}\sigma_1^2) - \frac{\alpha\beta SC}{I} - \frac{b_1\mu}{S} - \frac{b_2 q\gamma_1 I}{C} + b_1(\mu + \frac{1}{2}\sigma_1^2) \\ &\quad + b_2(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2) + \gamma_1 + \mu + \frac{1}{2}\sigma_2^2 + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q\gamma_1)}{\gamma_2 + \mu} I \\ &\leq -2\sqrt{\beta a_1\mu} + a_1(\mu + \frac{1}{2}\sigma_1^2) - 3\sqrt[3]{\alpha\beta q\gamma_1\mu b_1 b_2} + b_1(\mu + \frac{1}{2}\sigma_1^2) \\ &\quad + b_2(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2) + \gamma_1 + \mu + \frac{1}{2}\sigma_2^2 + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q\gamma_1)}{\gamma_2 + \mu} I. \end{aligned}$$

Let

$$a_1 = \frac{\beta\mu}{(\mu + \frac{1}{2}\sigma_1^2)^2}, b_1 = \frac{\alpha\beta q\gamma_1\mu}{(\mu + \frac{1}{2}\sigma_1^2)^2(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)}, b_2 = \frac{\alpha\beta q\gamma_1\mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)^2}.$$

Then we have

$$\begin{aligned} LV_1 &\leq -\frac{\beta\mu}{\mu + \frac{1}{2}\sigma_1^2} - \frac{\alpha\beta q\gamma_1\mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)} + \gamma_1 + \mu + \frac{1}{2}\sigma_2^2 + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q\gamma_1)}{\gamma_2 + \mu} I \\ &= -(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(R_0^S - 1) + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q\gamma_1)}{\gamma_2 + \mu} I \\ &=: -\lambda + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q\gamma_1)}{\gamma_2 + \mu} I. \end{aligned} \tag{9}$$

Where

$$\lambda = (\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(R_0^S - 1) > 0.$$

Also, we have

$$LV_2 = -\frac{\mu}{S} - \frac{q\gamma_1 I}{C} + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)}{\gamma_2 + \mu} I + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2}. \tag{10}$$

$$\begin{aligned} LV_3 &= (S + I + C)^0 [\mu - \mu(S + I + C) - (\gamma_1 - q\gamma_1)I - \gamma_2 C] \\ &\quad + \frac{\theta}{2}(S + I + C)^{\theta-1} (\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 C^2) \\ &\leq \mu(S + I + C)^0 - \mu(S + I + C)^{\theta+1} + \frac{\theta}{2}(S + I + C)^{\theta-1} (\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 C^2) \\ &\leq \mu(S + I + C)^0 - [\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S + I + C)^{\theta+1} \\ &\leq B - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S + I + C)^{\theta+1} \\ &\leq B - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S^{\theta+1} + I^{\theta+1} + C^{\theta+1}), \end{aligned} \tag{11}$$

Where

$$B = \sup_{(S,I,C) \in R_+^3} \{ \mu(S + I + C)^0 - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S + I + R)^{\theta+1} \} < \infty.$$

Defining a C^2 - function $\hat{V}(S, I, C) : R_+^3 \rightarrow R$ as follows

$$\hat{V}(S, I, C) = MV_1 + V_2 + V_3.$$

where M is positive constants to be determined later.

It is obvious to see that

$$\liminf_{k \rightarrow \infty, (S,I,C) \in R_+^3 \setminus U_k} \hat{V}(S, I, C) = +\infty,$$

at which $U_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$. So, $\hat{V}(S, I, C)$ is a continuous function which processes a minimum point

$(\bar{S}_0, \bar{I}_0, \bar{C}_0)$ in the interior of R_+^3 .

Therefore, we can define a non-negative C^2 - function $V(S, I, C) : R_+^3 \rightarrow R$ as follows

$$V(S, I, C) = \hat{V}(S, I, C) - \hat{V}(\bar{S}_0, \bar{I}_0, \bar{C}_0).$$

From (9), (10) and (11), we have that

$$\begin{aligned} LV &\leq -M\lambda - \frac{\mu}{S} - \frac{q\gamma_1 I}{C} + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + \gamma_2 + 2\mu \\ &\quad + \frac{\sigma_1^2 + \sigma_3^2}{2} + B - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S^{\theta+1} + I^{\theta+1} + C^{\theta+1}) \\ &=: -M\lambda - \frac{\mu}{S} - \frac{q\gamma_1 I}{C} + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + Q(S, I, C), \end{aligned}$$

where

$$Q(S, I, C) = \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} + B - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S^{\theta+1} + I^{\theta+1} + C^{\theta+1}).$$

Let M is a positive constant and satisfy

$$-M\lambda + F \leq -2, \tag{12}$$

where

$$F = \sup_{(S, I, C) \in R_+^3} Q(S, I, C) < \infty.$$

Now, we define a bounded closed set as

$$D_\varepsilon = \{(S, I, C) \in R_+^3 : \varepsilon \leq S \leq \frac{1}{\varepsilon}, \varepsilon \leq I \leq \frac{1}{\varepsilon}, \varepsilon^2 \leq C \leq \frac{1}{\varepsilon^2}\}.$$

at which $\varepsilon > 0$ is a sufficiently small positive constant according with the following inequalities

$$-\frac{\mu}{\varepsilon} + U \leq -1. \tag{13}$$

$$-M\lambda + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} \varepsilon + F \leq -1. \tag{14}$$

$$-\frac{q\gamma_1}{\varepsilon} + U \leq -1. \tag{15}$$

$$-\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]\frac{1}{\varepsilon^{\theta+1}} + G \leq -1. \tag{16}$$

$$-\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]\frac{1}{\varepsilon^{\theta+1}} + H \leq -1. \tag{17}$$

$$-\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]\frac{1}{\varepsilon^{2(\theta+1)}} + N \leq -1. \tag{18}$$

Therefore, $R_+^3 \setminus D_\varepsilon$ can be divided into six domains

$$\begin{aligned} D_1 &= \{(S, I, C) \in R_+^3, 0 < S < \varepsilon\}, D_2 = \{(S, I, C) \in R_+^3, 0 < I < \varepsilon\}, \\ D_3 &= \{(S, I, C) \in R_+^3, 0 < C < \varepsilon^2, I \geq \varepsilon\}, D_4 = \{(S, I, C) \in R_+^3, S > \frac{1}{\varepsilon}\}, \\ D_5 &= \{(S, I, C) \in R_+^3, I > \frac{1}{\varepsilon}\}, D_6 = \{(S, I, C) \in R_+^3, C > \frac{1}{\varepsilon^2}\}. \end{aligned}$$

In the following we will illustrate $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in R_+^3 \setminus D_\varepsilon$.

Case 1. If $(S, I, C) \in D_1$, then we have that

$$\begin{aligned} LV &\leq -M\lambda - \frac{\mu}{S} - \frac{q\gamma_1 I}{C} + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + Q(S, I, C) \\ &\leq -\frac{\mu}{\varepsilon} + U. \end{aligned} \tag{19}$$

where

$$U = \sup_{(S, I, C) \in R_+^3} \left\{ \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + Q(S, I, C) \right\} < \infty. \tag{20}$$

It is easy for us to see that $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in D_1$ by means of (13).

Case 2. If $(S, I, C) \in D_2$, then we get that

$$\begin{aligned} LV &\leq -M\lambda - \frac{\mu}{S} + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + Q(S, I, C) \\ &\leq -M\lambda + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + F \\ &\leq -M\lambda + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} \varepsilon + F. \end{aligned} \quad (21)$$

According to (14), the conclusion that $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in D_2$ is true.

Case 3. If $(S, I, C) \in D_3$, then we see that

$$\begin{aligned} LV &\leq -\frac{q\gamma_1 I}{C} + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + Q(S, I, C) \\ &\leq -\frac{q\gamma_1}{C} + U. \end{aligned} \quad (22)$$

By means of (15), we obtain the conclusion that $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in D_3$.

Case 4. If $(S, I, C) \in D_4$, then we attain that

$$\begin{aligned} LV &\leq -\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]S^{\theta+1} - \frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]S^{\theta+1} + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} \\ &\quad + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + B - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](I^{\theta+1} + C^{\theta+1}) \\ &\leq -\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]S^{\theta+1} + G \\ &\leq -\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]\frac{1}{\varepsilon^{\theta+1}} + G. \end{aligned} \quad (23)$$

where

$$\begin{aligned} G &= \sup_{(S, I, C) \in \mathbb{R}_+^3} \left\{ -\frac{1}{4}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)]S^{\theta+1} + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} \right. \\ &\quad \left. + \frac{\beta(\gamma_2 + \mu + \alpha q\gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + B - \frac{1}{2}[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](I^{\theta+1} + C^{\theta+1}) \right\} < \infty. \end{aligned} \quad (24)$$

It is obvious that $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in D_4$ in virtue of (16).

Case 5. If $(S, I, C) \in D_5$, then we arrive at

$$\begin{aligned}
LV &\leq -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]I^{\theta+1} - \frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]I^{\theta+1} + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} \\
&\quad + \frac{\beta(\gamma_2 + \mu + \alpha q \gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + B - \frac{1}{2}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{\theta+1} + C^{\theta+1}) \\
&\leq -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]I^{\theta+1} + H \\
&\leq -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\varepsilon^{\theta+1}} + H.
\end{aligned} \tag{25}$$

Where

$$\begin{aligned}
H = \sup_{(S,I,C) \in R_+^3} \left\{ -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]I^{\theta+1} + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} \right. \\
\left. + \frac{\beta(\gamma_2 + \mu + \alpha q \gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + B - \frac{1}{2}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{\theta+1} + C^{\theta+1}) \right\} < \infty.
\end{aligned} \tag{26}$$

With the aid of (17), the result that $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in D_5$ is right.

Case 6. If $(S, I, C) \in D_6$, then we have a conclusion as follows

$$\begin{aligned}
LV &\leq -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]C^{\theta+1} - \frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]C^{\theta+1} + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} \\
&\quad + \frac{\beta(\gamma_2 + \mu + \alpha q \gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + B - \frac{1}{2}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{\theta+1} + I^{\theta+1}) \\
&\leq -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]C^{\theta+1} + N \\
&\leq -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\varepsilon^{2(\theta+1)}} + N.
\end{aligned} \tag{27}$$

Where

$$\begin{aligned}
N = \sup_{(S,I,C) \in R_+^3} \left\{ -\frac{1}{4}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]C^{\theta+1} + \gamma_2 + 2\mu + \frac{\sigma_1^2 + \sigma_3^2}{2} \right. \\
\left. + \frac{\beta(\gamma_2 + \mu + \alpha q \gamma_1)(Ma_1 + Mb_1 + 1)}{\gamma_2 + \mu} I + B - \frac{1}{2}\left[\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{\theta+1} + I^{\theta+1}) \right\} < \infty.
\end{aligned} \tag{28}$$

where U, G, H, N are all positive constant according with the equations (20),(24),(26),(28). We draw the conclusion that $LV(S, I, C) \leq -1$ for any given $(S, I, C) \in D_6$ by feat of (18).

From the discussion of a series of cases above, we come to a conclusion that

$$LV(S, I, C) \leq -1, (S, I, R) \in R_+^3 \setminus D_\varepsilon.$$

Next, the diffusion matrix of system (3) is

$$A = \begin{pmatrix} \sigma_1^2 S^2 & 0 & 0 \\ 0 & \sigma_2^2 I^2 & 0 \\ 0 & 0 & \sigma_3^2 C^2 \end{pmatrix}$$

Let $c = \min_{(S,I,C) \in \bar{D}_k \subset R_+^3} \{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 C^2\}$, then we can have

$$\sum_{i,j=1}^3 a_{ij}(x) \xi_i \xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 + \sigma_3^2 C^2 \xi_3^2 \geq c |\xi|^2, (S, I, C) \in \bar{D}_k, \xi \in R^3.$$

Hence the condition (H1) in Lemma 1 is verified successfully.

As a result, according to Lemma 1, we get that there exists a unique ergodic stationary distribution. The proof is finished.

4. THE EXTINCTION AND PERSISTENCE OF SYSTEM (3)

For convenience, we introduce the following notation and lemma.

$$\langle f(t) \rangle = \frac{1}{t} \int_0^t f(\zeta) d\zeta, \text{ herein } f(t) \text{ is an integral function on } [0, +\infty).$$

Lemma 2. (Strong law of large number) (see [38]). Let $M = \{M\}_{t \geq 0}$ be a real valued continuous local martingale which disappear at $t = 0$. Then

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty \text{ a.s.} \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0 \text{ a.s.}$$

Also

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty \text{ a.s.} \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0 \text{ a.s.}$$

Lemma 3. Assume that $(S(t), I(t), C(t))$ be the solution of the system (3) having the initial value $(S(0), I(0), C(0)) \in R_+^3$, then we get

$$\limsup_{t \rightarrow \infty} (S(t) + I(t) + C(t)) < \infty \text{ a.s.}$$

as well as

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{I(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{C(t)}{t} = 0, \text{ a.s.} \tag{29}$$

$$\lim_{t \rightarrow \infty} \frac{\ln S(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\ln I(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\ln C(t)}{t} = 0, \text{ a.s.} \tag{30}$$

and

$$\lim_{t \rightarrow \infty} \frac{\int_0^t S(\zeta) dB_1(\zeta)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\int_0^t I(\zeta) dB_2(\zeta)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\int_0^t C(\zeta) dB_3(\zeta)}{t} = 0. \text{ a.s.} \tag{31}$$

Proof. We have the following equation via the system (3)

$$d(S + I + C) = [\mu - \mu(S + I + C) - (1 - q)\gamma_1 I - \gamma_2 C] dt + \sigma_1 S dB_1(t) + \sigma_2 I dB_2(t) + \sigma_3 C dB_3(t). \tag{32}$$

By solving equation (32), we can get

$$\begin{aligned}
 S(t) + I(t) + C(t) &= 1 + (S(0) + I(0) + C(0) - 1)e^{-\mu t} - (1 - q)\gamma_1 \int_0^t e^{-\mu(t-\zeta)} I(\zeta) d(\zeta) \\
 &\quad - \gamma_2 \int_0^t e^{-\mu(t-\zeta)} C(\zeta) d(\zeta) + \sigma_1 \int_0^t e^{-\mu(t-\zeta)} S(\zeta) dB_1(\zeta) \\
 &\quad + \sigma_2 \int_0^t e^{-\mu(t-\zeta)} I(\zeta) dB_2(\zeta) + \sigma_3 \int_0^t e^{-\mu(t-\zeta)} C(\zeta) dB_3(\zeta) \\
 &\leq 1 + (S(0) + I(0) + C(0) - 1)e^{-\mu t} + N(t). \quad a.s.
 \end{aligned}
 \tag{33}$$

Where

$$N(t) = \sigma_1 \int_0^t e^{-\mu(t-\zeta)} S(\zeta) dB_1(\zeta) + \sigma_2 \int_0^t e^{-\mu(t-\zeta)} I(\zeta) dB_2(\zeta) + \sigma_3 \int_0^t e^{-\mu(t-\zeta)} C(\zeta) dB_3(\zeta).$$

It is apparent that $N(t)$ is a continuous local martingale having the property of $N(0) = 0$.

Let

$$X(t) = X(0) + W(t) - P(t) + N(t).$$

at which

$$X(0) = S(0) + I(0) + C(0), W(t) = 1 - e^{-\mu t}, P(t) = [S(0) + I(0) + C(0)](1 - e^{-\mu t}). \tag{34}$$

By means of (34), we can get that

$$S(t) + I(t) + C(t) \leq X(t) \quad a.s. \quad t \geq 0.$$

From the expression of $W(t)$ and $P(t)$, it is easy for us to see that $W(t)$ and $P(t)$ are continuous increasing course processing the property of $W(0) = P(0) = 0$ for all $t \geq 0$. So we can attain that $\lim_{t \rightarrow \infty} X(t) < \infty$ a.s. According to

Theorem 3.9 in the literature [44, 45]. Therefore

$$\limsup_{t \rightarrow \infty} (S(t) + I(t) + C(t)) < \infty \quad a.s.$$

Let

$$N_1(t) = \int_0^t S(\zeta) d(\zeta), N_2(t) = \int_0^t I(\zeta) d(\zeta), N_3(t) = \int_0^t C(\zeta) d(\zeta).$$

Then we can obtain

$$\limsup_{t \rightarrow \infty} \frac{\langle N_1, N_1 \rangle_t}{t} \leq \sup_{t \geq 0} S^2(t) < \infty.$$

So according to Lemma 2, we have $\lim_{t \rightarrow \infty} \frac{\int_0^t S(\zeta) dB_1(\zeta)}{t} = 0$. a.s. In a similar way, we can attain

$$\lim_{t \rightarrow \infty} \frac{\int_0^t I(\zeta) dB_2(\zeta)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\int_0^t C(\zeta) dB_3(\zeta)}{t} = 0. \quad a.s.$$

Next because $S(t), I(t), C(t)$ are bounded and positive, we have the conclusion

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{I(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{C(t)}{t} = 0, \quad a.s.$$

and

$$\lim_{t \rightarrow \infty} \frac{\ln S(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\ln I(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\ln C(t)}{t} = 0, \quad a.s.$$

This finishes the proof of Lemma 3.

4.1 Disease Extinction

In this subsection, we derive in detail the condition for the extinction of the disease.

Theorem 3. Assume that $(S(t), I(t), C(t))$ is the solution of the system (3) having the initial value $(S(0), I(0), C(0)) \in R_+^3$. If

$$\bar{R}_0 = \frac{6\beta(\gamma_1 + \mu)(\gamma_1 + \mu + \alpha q\gamma_1)}{(\gamma_1 + \mu)^2(\gamma_2 + \mu + \frac{\sigma_3^2}{2}) \wedge \frac{\sigma_2^2 q^2 \gamma_1^2}{2}} < 1,$$

then

$$\limsup_{t \rightarrow \infty} \frac{\ln[q\gamma_1 I + (\gamma_1 + \mu)C]}{t} \leq \frac{[(\gamma_1 + \mu)^2(\gamma_2 + \mu + \frac{\sigma_3^2}{2}) \wedge \frac{\sigma_2^2 q^2 \gamma_1^2}{2}]}{2(\gamma_1 + \mu)^2} (\bar{R}_0 - 1) < 0.$$

as well as

$$\limsup_{t \rightarrow \infty} \langle S(t) \rangle = 1 \text{ . a.s.}$$

Proof. Let $V_0(t) = \ln[q\gamma_1 I(t) + (\gamma_1 + \mu)C(t)]$, applying the Itô formula to $V_0(t)$, then we have

$$\begin{aligned} dV_0(t) &= \left\{ \frac{q\gamma_1\beta SI + q\gamma_1\alpha\beta SC - (\gamma_1 + \mu)(\gamma_2 + \mu)C}{q\gamma_1 I + (\gamma_1 + \mu)C} - \frac{q^2\gamma_1^2\sigma_2^2 I^2 + (\gamma_1 + \mu)^2\sigma_3^2 C^2}{2[q\gamma_1 I + (\gamma_1 + \mu)C]^2} \right\} dt \\ &+ \frac{\sigma_2 q\gamma_1 I}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_2(t) + \frac{\sigma_3(\gamma_1 + \mu)C}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_3(t) \\ &\leq \frac{\beta(\gamma_1 + \mu + \alpha q\gamma_1)S}{\gamma_1 + \mu} dt - \left[\frac{(\gamma_1 + \mu)(\gamma_2 + \mu)C}{q\gamma_1 I + (\gamma_1 + \mu)C} + \frac{q^2\gamma_1^2\sigma_2^2 I^2 + (\gamma_1 + \mu)^2\sigma_3^2 C^2}{2[q\gamma_1 I + (\gamma_1 + \mu)C]^2} \right] dt \\ &+ \frac{\sigma_2 q\gamma_1 I}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_2(t) + \frac{\sigma_3(\gamma_1 + \mu)C}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_3(t) \\ &\leq \frac{\beta(\gamma_1 + \mu + \alpha q\gamma_1)S}{\gamma_1 + \mu} dt - \frac{[(\gamma_1 + \mu)^2(\gamma_2 + \mu + \frac{\sigma_3^2}{2}) \wedge \frac{\sigma_2^2 q^2 \gamma_1^2}{2}](I^2 + C^2)}{[q\gamma_1 I + (\gamma_1 + \mu)C]^2} dt \\ &+ \frac{\sigma_2 q\gamma_1 I}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_2(t) + \frac{\sigma_3(\gamma_1 + \mu)C}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_3(t) \\ &\leq \frac{\beta(\gamma_1 + \mu + \alpha q\gamma_1)S}{\gamma_1 + \mu} dt - \frac{1}{2(\gamma_1 + \mu)^2} [(\gamma_1 + \mu)^2(\gamma_2 + \mu + \frac{\sigma_3^2}{2}) \wedge \frac{\sigma_2^2 q^2 \gamma_1^2}{2}] dt \\ &+ \frac{\sigma_2 q\gamma_1 I}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_2(t) + \frac{\sigma_3(\gamma_1 + \mu)C}{q\gamma_1 I + (\gamma_1 + \mu)C} dB_3(t) \end{aligned}$$

In addition, according to the system (3), we can get

$$\begin{aligned} d(S + I + C) &= [\mu - \mu(S + I + C) - (\gamma_1 - q\gamma_1)I - \gamma_2 C] dt \\ &+ \sigma_1 S dB_1(t) + \sigma_2 I dB_2(t) + \sigma_3 C dB_3(t) \\ &\leq [\mu - \mu(S + I + C)] dt + \sigma_1 S dB_1(t) + \sigma_2 I dB_2(t) + \sigma_3 C dB_3(t). \end{aligned} \tag{35}$$

Integrating both sides of (35) from 0 to t and dividing by t , then taking the limit superior both sides of (35), finally by means of (33) in Lemma 3, we can obtain

$$\limsup_{t \rightarrow \infty} \langle S(t) + I(t) + C(t) \rangle \leq 1 \text{ . a.s.}$$

Therefore, from the inequation, we have the conclusion that

$$\limsup_{t \rightarrow \infty} \langle S(t) \rangle \leq 3 \text{ . a.s.} \tag{36}$$

Integrating both sides of (34) from 0 to t and dividing by t , then taking the limit superior both sides of (34). Further according to (4.8) and the assumption $\bar{R}_0 < 1$, we have the conclusion

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln[q\gamma_1 I + (\gamma_1 + \mu)C]}{t} &\leq \frac{3\beta(\gamma_1 + \mu + \alpha q\gamma_1)}{\gamma_1 + \mu} - \frac{1}{2(\gamma_1 + \mu)^2} [(\gamma_1 + \mu)^2(\gamma_2 + \mu + \frac{\sigma_3^2}{2}) \wedge \frac{\sigma_2^2 q^2 \gamma_1^2}{2}] \\ &= \frac{(\gamma_1 + \mu)^2(\gamma_2 + \mu + \frac{\sigma_3^2}{2}) \wedge \frac{\sigma_2^2 q^2 \gamma_1^2}{2}}{2(\gamma_1 + \mu)^2} (\bar{R}_0 - 1) < 0. \end{aligned} \tag{37}$$

From (37), it is obvious that

$$\lim_{t \rightarrow \infty} I(t) = 0, \lim_{t \rightarrow \infty} C(t) = 0 \text{ . a.s.} \tag{38}$$

Next, we take the integration of both sides of the system (3), then we obtain the following result

$$\begin{aligned} \frac{S(t) - S(0)}{t} &= \mu - \beta \langle S(t)I(t) \rangle - \alpha\beta \langle S(t)C(t) \rangle - \mu \langle S(t) \rangle + \sigma_1 \frac{\int_0^t S(\zeta)dB_1(\zeta)}{t} \\ \frac{I(t) - I(0)}{t} &= \beta \langle S(t)I(t) \rangle + \alpha\beta \langle S(t)C(t) \rangle - (\gamma_1 + \mu) \langle I(t) \rangle + \sigma_2 \frac{\int_0^t I(\zeta)dB_2(\zeta)}{t} \\ \frac{C(t) - C(0)}{t} &= q\gamma_1 \langle I(t) \rangle - (\gamma_2 + \mu) \langle C(t) \rangle + \sigma_3 \frac{\int_0^t C(\zeta)dB_3(\zeta)}{t}. \end{aligned} \tag{39}$$

Adding up the three equality of (39), which yields

$$\begin{aligned} \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{C(t) - C(0)}{t} &= \mu - \mu \langle S(t) \rangle - (\gamma_1 + \mu - q\gamma_1) \langle I(t) \rangle - (\gamma_2 + \mu) \langle C(t) \rangle \\ &\quad + \sigma_1 \frac{\int_0^t S(\zeta)dB_1(\zeta)}{t} + \sigma_2 \frac{\int_0^t I(\zeta)dB_2(\zeta)}{t} + \sigma_3 \frac{\int_0^t C(\zeta)dB_3(\zeta)}{t}. \end{aligned} \tag{40}$$

By (29),(31),(38),(40), we draw the conclusion as follows

$$\limsup_{t \rightarrow \infty} \langle S(t) \rangle = 1 \text{ . a.s.}$$

The proof is finished.

4.2 Persistence in mean

In this subsection, we will derive the condition for the persistence in mean of disease, at first, we introduce the definition of persistence in mean.

Definition The system (3) is defined as persistence in mean if

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(\zeta)d\zeta > 0 \text{ . a.s.}$$

Theorem 4. For given initial value $(S(0), I(0), C(0)) \in \Gamma$, if

$$R_0^S := \frac{\beta\mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)} + \frac{\alpha\beta q\gamma_1\mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)} > 1,$$

then

$$\liminf_{t \rightarrow \infty} \langle I(t) \rangle \geq \frac{(\gamma_2 + \mu)(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(R_0^S - 1)}{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q \gamma_1)} \text{ . a.s.}$$

Where

$$a_1 = \frac{\beta\mu}{(\mu + \frac{1}{2}\sigma_1^2)^2}, b_1 = \frac{\alpha\beta q \gamma_1 \mu}{(\mu + \frac{1}{2}\sigma_1^2)^2(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)}.$$

In other words, the disease will prevail if $R_0^S > 1$.

Proof. Taking advantage of the same function as the proof of Theorem 3.1, let

$$V_1(S, I, C) = -\ln I - (a_1 + b_1) \ln S - b_2 \ln C + \frac{\alpha\beta(a_1 + b_1)}{\gamma_2 + \mu} C.$$

where a_1, b_1, b_2 are the same expression as the proof of Theorem 3.1. Using the Itô formula to V_1 , we can attain that

$$dV_1 = LV_1 dt - \sigma_2 dB_2(t) - (a_1 + b_1)\sigma_1 dB_1(t) - b_2\sigma_3 dB_3(t) + \frac{\alpha\beta(a_1 + b_1)}{\gamma_2 + \mu} \sigma_3 C dB_3(t). \tag{41}$$

From the proof of Theorem 2, we can get

$$\begin{aligned} LV &\leq -\frac{\beta\mu}{\mu + \frac{1}{2}\sigma_1^2} - \frac{\alpha\beta q \gamma_1 \mu}{(\mu + \frac{1}{2}\sigma_1^2)(\gamma_2 + \mu + \frac{1}{2}\sigma_3^2)} + \gamma_1 + \mu + \frac{1}{2}\sigma_2^2 + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q \gamma_1)}{\gamma_2 + \mu} I \\ &= -(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(R_0^S - 1) + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q \gamma_1)}{\gamma_2 + \mu} I. \end{aligned}$$

(42)

Next, based on (42), we take the integration both sides of (41) from 0 to t and dividing by t , then we can attain

$$\begin{aligned} \frac{V_1(S(t), I(t), C(t)) - V_1(S(0), I(0), C(0))}{t} &\leq -(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(R_0^S - 1) \\ &\quad + \frac{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q \gamma_1)}{\gamma_2 + \mu} \langle I(t) \rangle - \frac{\phi(t)}{t}. \end{aligned} \tag{43}$$

Where

$$\phi(t) = \sigma_2 dB_2(t) + (a_1 + b_1)\sigma_1 dB_1(t) + b_2\sigma_3 dB_3(t) - \frac{\alpha\beta(a_1 + b_1)}{\gamma_2 + \mu} \sigma_3 C dB_3(t)$$

According Lemma 2 and (30) in Lemma 3, we can see that

$$\limsup_{t \rightarrow \infty} \frac{\phi(t)}{t} = 0 \text{ . a.s.}$$

Next using (30), (43) and taking advantage of the assumption that $R_0^S > 1$. Then by taking the limit inferior of both side of (43), we have

$$\liminf_{t \rightarrow \infty} \langle I(t) \rangle \geq \frac{(\gamma_2 + \mu)(\gamma_1 + \mu + \frac{1}{2}\sigma_2^2)(R_0^S - 1)}{\beta(a_1 + b_1)(\gamma_2 + \mu + \alpha q \gamma_1)}.$$

The proof is completed.

5. CONCLUSIONS AND FUTURE PROSPECTS

Aiming at the transmission characteristics of potato early and late blight, this paper constructs a stochastic SICK transmission model incorporating environmental noise and systematically analyzes the dynamical behaviors of the model. The results show that: the model admits a unique global positive solution, which ensures the biological rationality of the model; when $R_0^S > 1$, the model has an ergodic stationary distribution and the disease will persistently prevail; when $\bar{R}_0 < 1$, the disease will eventually die out. This study provides a theoretical basis for the epidemic prediction and prevention and control of potato early and late blight.

This study still has several limitations, and future research can be further carried out from the following aspects: Introduce the time-delay effect to explore the impacts of pathogen incubation period and disease detection delay on transmission dynamics;

Construct a stochastic model with periodic coefficients to match the seasonal fluctuation characteristics of environmental factors such as temperature and humidity;

Incorporate practical prevention and control measures (e.g., chemical control, cultivation of disease-resistant varieties, crop rotation, etc.) to establish a stochastic model with control strategies, so as to provide more direct theoretical support for formulating precise prevention and control schemes.

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