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Research Article

One-Step Four Point Hybrid Block Scheme Designed for the Solution of General Third Order Ordinary Differential Equations

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Abstract
In this paper, one-step four point hybrid block technique is formulated and employed to solve general third-order ordinary differential equations directly. The construction of the method utilized interpolation and collocation, with power series serving as the basis function. An analysis of its properties such as order, convergence, consistency, zero stability and stability region was examined. When tested on third order ODEs problems, the results indicated that the method produced superior performance compared to those in the literature.

Keywords: *One-step, off-grid point, basis function, Hybrid block formula, interpolation, collocation*, *Local Truncation Error.*

1. INTRODUCTION
Ordinary Differential Equations (ODEs) are commonly used in mathematically formulated models designed to describe physical phenomena across science and engineering disciplines. They play a crucial role and find wide-ranging applications, not only in the physical sciences but also in various other areas such as medical sciences, thermodynamics, chemical engineering, control theory, operations research, and behavioural sciences.

Let us examine the general third-order ordinary differential equations (ODEs) expressed using a sixth-order power series of the form:

$$y(x)=\sum_{j=0}^{k} b_j x^j \tag{1}$$

Which is recommend as general third order derivative solution of initial value problems of the form

$$y'''(x)=f(x,y,y',y''),\, y(x)=y_0,\, y'(0)=y'_0,\, y''(0)=y_0'' \tag{2}$$

The resolution of (1) has been explored by numerous scholars, such as: Abdulazeez, Kayode Jimoh [1] proposed two-step hybrid block method for the numerical solution third order differential equations. He adopted the used of approximate power series as an interpolation equation and its derivatives as a collocation equation that is used in the development of the method. Ishaq *et al.* [2] introduced a novel three-step block method designed to directly tackle third-order initial value problems through the method of collocation. Their method was zero-stable, convergent and the region of stability is absolutely stable. Dalatu *et al.* [3] developed a hybrid block method for solving third-order derivative with initial value problems of ordinary differential equations. Their method was derived by collocating and interpolating the approximate solution using power series. Abdelrahim and Omar [4] developed one-step blocks method for the direct solution of third-order initial value problems of ordinary differential equations using the power series as the basis function. Their method was developed to solve third-order initial value problems. Atabo *et al.* [5] developed a selected single step hybrid block formula for solving third-order ordinary differential equations with application in thin film flow. Their method has advantage of selecting only odd off-grid points within a single-step interval and collocated at all points. Modebei *et al.* [6] proposed a three-step fourth derivatives method for numerical integration of third order ordinary differential equations. They used a three-step hybrid block method with three mid-step grid points based on linear multistep method to presented in their work for direct approximation of solution of third-order initial and boundary value problems. Muhammed and Adeniyi [7] developed three-step implicit hybrid linear method for solution of third-order ordinary differential equations. Adeyeye and Omar [8] developed third-order ordinary differential equations using one-step block method with four equidistant generalized hybrid points. The equation for the generalized linear block method takes a similar form as the conventional linear multistep method, however the form produces the needed family of scheme required simultaneously evaluate the solution of the third-order ordinary differential equations at individual grid points in a self-starting mode. Joshua, S. [9] a hybrid block technique with two-step optimization for handling general third order ordinary differential equations.

2. Derivation of the Method

The derivation of the one-step third derivative method is based on a finite power series function expressed as:

$$y(x)=\sum_{j=0}^{(l+c)-1} b_j x^j$$

Which is propose as general third order derivative solution of initial value problems of the form

$$y'''(x)=f(x,y,y',y''),\, y(x)=y_0,\, y'(0)=y'_0,\, y''(0)=y_0''$$

Where i and c denote points of interpolation and collocation respectively, so that the third derivative is

$$y'''(x)=\sum_{j=0}^{(l+c)-1} j(j-1)(j-2)b_j x^{j-3} \tag{3}$$

Interpolating (1) at $x_{n+i}, i=0,\frac{1}{9},\frac{1}{3}$ and collocating at $x_{n+r}, r=0,\frac{1}{9},\frac{1}{3},\frac{5}{9},\frac{7}{9},1$ we obtained a system of nonlinear equations of the form

$$AX=U \tag{4}$$

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
1 & x_{n+\frac{1}{9}} & x^2_{n+\frac{1}{9}} & x^3_{n+\frac{1}{9}} & x^4_{n+\frac{1}{9}} & x^5_{n+\frac{1}{9}} & x^6_{n+\frac{1}{9}} & x^7_{n+\frac{1}{9}} & x^8_{n+\frac{1}{9}} \\
1 & x_{n+\frac{1}{3}} & x^2_{n+\frac{1}{3}} & x^3_{n+\frac{1}{3}} & x^4_{n+\frac{1}{3}} & x^5_{n+\frac{1}{3}} & x^6_{n+\frac{1}{3}} & x^7_{n+\frac{1}{3}} & x^8_{n+\frac{1}{3}} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{8}{3}} & 60x^2_{n+\frac{1}{9}} & 120x^3_{n+\frac{1}{9}} & 210x^4_{n+\frac{1}{9}} & 336x^5_{n+\frac{1}{9}} \\
0 & 0 & 0 & 6 & 24x_{n+8} & 60x^2_{n+\frac{1}{3}} & 120x^2_{n+\frac{1}{3}} & 210x^2_{n+\frac{1}{3}} & 336x^2_{n+\frac{1}{3}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{40}{3}} & 60x^2_{n+\frac{5}{9}} & 120x^2_{n+\frac{5}{9}} & 210x^2_{n+\frac{5}{9}} & 336x^2_{n+\frac{5}{9}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{56}{3}} & 60x^2_{n+\frac{7}{9}} & 120x^2_{n+\frac{7}{9}} & 210x^2_{n+\frac{7}{9}} & 336x^2_{n+\frac{7}{9}} \\
0 & 0 & 0 & 6 & 24x_{n+24} & 60x^2_{n+1} & 120x^2_{n+1} & 210x^2_{n+1} & 336x^2_{n+1}
\end{pmatrix}$$

$$X=[b_0,b_1,b_3,b_4,b_5,b_6,b_7,b_8]^T, U=\left[y_n, y_{n+\frac{1}{9}}, y_{n+\frac{1}{3}}, f_n, f_{n+\frac{1}{9}}, f_{n+\frac{1}{3}}, f_{n+\frac{5}{9}}, f_{n+\frac{7}{9}}, f_{n+1}\right]^T$$

Whose unknowns $b's$ are solved for using Gaussian elimination technique and results are substituted in to equation (1) to give a continuous linear multistep method of the form

$$y(x)=\lambda_0 y_n+\lambda_{\frac{1}{9}} y_{n+\frac{1}{9}}+\lambda_{\frac{1}{3}} y_{n+\frac{1}{3}}+h^3\left[\phi_0 f_n+\phi_{\frac{1}{9}} f_{n+\frac{1}{9}}+\phi_{\frac{1}{3}} f_{n+\frac{1}{3}}+\phi_{\frac{5}{9}} f_{n+\frac{5}{9}}+\phi_{\frac{7}{9}} f_{n+\frac{7}{9}}+\phi_1 f_{n+1}\right] \quad (5)$$

The expression for the $\lambda's$ then written in terms of the parameters $\lambda_j's$ and $\phi_j's$ as the following functions of t.

$$\lambda_0(t)=27t^2-12t+1$$

$$\lambda_{\frac{1}{9}}(t)=\frac{27}{2}t-\frac{81}{2}t^2$$

$$\lambda_{\frac{1}{3}}(t)=-\frac{3}{2}t+\frac{27}{2}t^2$$

$$\phi_0(t)=\frac{1553}{2143260}th^3-\frac{10525}{571536}t^2h^3+\frac{1}{6}t^3h^3-\frac{563}{840}t^4h^3+\frac{19}{14}t^5h^3-\frac{207}{140}t^6h^3+\frac{81}{98}t^7h^3-\frac{729}{3920}t^8h^3$$

$$\phi_{\frac{1}{9}}(t)=\frac{2437}{483840}th^3-\frac{34579}{645120}t^2h^3+\frac{945}{1024}t^4h^3-\frac{837}{320}t^5h^3+\frac{8343}{2560}t^6h^3-\frac{2187}{1120}t^7h^3+\frac{6561}{14336}t^8h^3$$

$$\phi_{\frac{1}{3}}(t)=\frac{1537}{3265920}th^3-\frac{7093}{4354560}t^2h^3-\frac{105}{256}t^4h^3+\frac{687}{320}t^5h^3-\frac{1107}{320}t^6h^3+\frac{2673}{1120}t^7h^3-\frac{2187}{3584}t^8h^3$$

$$\phi_{\frac{5}{9}}(t)=-\frac{17}{241920}th^3-\frac{229}{322560}t^2h^3+\frac{567}{2560}t^4h^3-\frac{81}{64}t^5h^3+\frac{3159}{1280}t^6h^3-\frac{2187}{1120}t^7h^3+\frac{19683}{35840}t^8h^3$$

$$\phi_{\frac{7}{9}}(t)=\frac{31}{2540160}th^3+\frac{229}{677376}t^2h^3-\frac{135}{1792}t^4h^3+\frac{999}{2240}t^5h^3-\frac{1053}{1120}t^6h^3+\frac{6561}{7840}t^7h^3-\frac{6561}{25088}t^8h^3$$

$$\phi_1(t)=-\frac{43}{39191040}th^3-\frac{2987}{52254720}t^2h^3+\frac{35}{3072}t^4h^3-\frac{11}{160}t^5h^3+\frac{387}{2560}t^6h^3-\frac{81}{560}t^7h^3+\frac{729}{14336}t^8h^3$$

The parameters $\lambda_j(t),\phi_j(t)$ are evaluated at $t=0,\frac{1}{9},\frac{1}{3},\frac{5}{9},\frac{7}{9},1$. The values obtained are then substituted in to (5) to obtain the implicit hybrid block method as follows

$$y_{n+\frac{5}{9}}=\frac{8}{3}y_n-5y_{n+\frac{1}{9}}+\frac{10}{3}y_{n+\frac{1}{3}}-\frac{319}{826686}h^3f_n+\frac{2137}{559872}h^3f_{n+\frac{1}{9}}+\frac{7609}{1259712}h^3f_{n+\frac{1}{3}}-\frac{137}{279936}h^3f_{n+\frac{5}{9}}+\frac{187}{979776}h^3f_{n+\frac{7}{9}}-\frac{421}{15116544}h^3f_{n+1} \quad (6)$$

$$y_{n+\frac{7}{9}} = 8y_n - 14y_{n+\frac{1}{9}} + 7y_{n+\frac{1}{3}} - \frac{163}{153090}h^3 f_n + \frac{31543}{2799360}h^3 f_{n+\frac{1}{9}} + \frac{16607}{699840}h^3 f_{n+\frac{1}{3}} + \frac{125}{31104}h^3 f_{n+\frac{5}{9}} + \frac{2549}{4898880}h^3 f_{n+\frac{7}{9}} - \frac{59}{933120}h^3 f_{n+1} \quad (7)$$

$$y_{n+1} = 16y_n - 27y_{n+\frac{1}{9}} + 12y_{n+\frac{1}{3}} - \frac{163}{76545}h^3 f_n + \frac{391}{17280}h^3 f_{n+\frac{1}{9}} + \frac{6133}{116640}h^3 f_{n+\frac{1}{3}} + \frac{169}{8640}h^3 f_{n+\frac{5}{9}} + \frac{559}{90720}h^3 f_{n+\frac{7}{9}} - \frac{49}{1399680}h^3 f_{n+1} \quad (8)$$

Differentiating (5) once, we have

$$hy'(x) = \lambda_0' y_n + \lambda_{\frac{1}{9}}' y_{n+\frac{1}{9}} + \lambda_{\frac{1}{3}}' y_{n+\frac{1}{3}} + h^3\left[\phi_0' f_n + \phi_{\frac{1}{9}}' f_{n+\frac{1}{9}} + \phi_{\frac{1}{3}}' f_{n+\frac{1}{3}} + \phi_{\frac{5}{9}}' f_{n+\frac{5}{9}} + \phi_{\frac{7}{9}}' f_{n+\frac{7}{9}} + \phi_1' f_{n+1}\right] \quad (9)$$

The derivatives of the parameters λ_j's and ϕ_j's are written as the following function of t.

$$\lambda'_0(t) = 54t - 12$$

$$\lambda'_{\frac{1}{9}}(t) = \frac{27}{2} - 81t$$

$$\lambda'_{\frac{1}{3}}(t) = -\frac{3}{2} + 27t$$

$$\phi'_0(t) = \frac{1553}{2143260}h^3 - \frac{10525}{285768}th^3 + \frac{1}{2}t^2h^3 - \frac{563}{210}t^3h^3 + \frac{95}{14}t^4h^3 - \frac{621}{70}t^5h^3 + \frac{81}{14}t^6h^3 - \frac{729}{490}t^7h^3$$

$$\phi'_{\frac{1}{9}}(t) = \frac{2437}{483840}h^3 - \frac{34579}{322560}th^3 + \frac{945}{256}t^3h^3 - \frac{837}{64}t^4h^3 + \frac{25029}{1280}t^5h^3 - \frac{2187}{160}t^6h^3 + \frac{6561}{1792}t^7h^3$$

$$\phi'_{\frac{1}{3}}(t) = \frac{1537}{3265920}h^3 - \frac{7093}{2177280}th^3 - \frac{105}{64}t^3h^3 + \frac{687}{64}t^4h^3 - \frac{3321}{160}t^5h^3 + \frac{2673}{160}t^6h^3 - \frac{2187}{448}t^7h^3$$

$$\phi'_{\frac{5}{9}}(t) = -\frac{17}{241920}h^3 - \frac{229}{161280}th^3 + \frac{567}{640}t^3h^3 - \frac{405}{64}t^4h^3 + \frac{9477}{640}t^5h^3 - \frac{2187}{160}t^6h^3 + \frac{19683}{4480}t^7h^3$$

$$\phi'_{\frac{7}{9}}(t) = \frac{31}{2540160}h^3 + \frac{229}{338688}th^3 - \frac{135}{448}t^3h^3 + \frac{999}{448}t^4h^3 - \frac{3159}{560}t^5h^3 + \frac{6561}{1120}t^6h^3 - \frac{6561}{3136}t^7h^3$$

$$\phi'_1(t) = -\frac{43}{39191040}h^3 - \frac{2987}{26127360}t h^3 + \frac{35}{768}t^3 h^3 - \frac{11}{32}t^4 h^3 + \frac{1161}{1280}t^5 h^3 - \frac{81}{80}t^6 h^3 + \frac{729}{1792}t^7 h^3$$

The derivatives $\lambda_j{}'s, \phi_j{}'s$ at $t = 0, \frac{1}{9}, \frac{1}{3}, \frac{5}{9}, \frac{7}{9}, 1$ the values obtained are then substituted in to (9) to obtain the following implicit hybrid block scheme

$$hy'_n = -12y_n + \frac{27}{2}y_{n+\frac{1}{9}} - \frac{3}{2}y_{n+\frac{1}{3}} + \frac{1553}{2143260}h^3 f_n + \frac{19}{857304}h^3 f_{n+\frac{1}{9}} - \frac{659}{857304}h^3 f_{n+\frac{1}{3}} - \frac{163}{68040}h^3 f_{n+\frac{5}{9}} - \frac{17243}{4286520}h^3 f_{n+\frac{7}{9}} - \frac{20633}{4286520}h^3 f_{n+1} \quad (10)$$

$$hy'_{n+\frac{1}{9}} = -6y_n + \frac{9}{2}y_{n+\frac{1}{9}} + \frac{3}{2}y_{n+\frac{1}{3}} + \frac{2437}{483840}h^3 f_n - \frac{91393}{26127360}h^3 f_{n+\frac{1}{9}} + \frac{7699}{967680}h^3 f_{n+\frac{1}{3}} + \frac{663043}{26127360}h^3 f_{n+\frac{5}{9}} + \frac{1112837}{26127360}h^3 f_{n+\frac{7}{9}} + \frac{1591}{27648}h^3 f_{n+1} \quad (11)$$

$$hy'_{n+\frac{1}{3}} = 6y_n - \frac{27}{2}y_{n+\frac{1}{9}} + \frac{15}{2}y_{n+\frac{1}{3}} + \frac{1537}{3265920}h^3 f_n - \frac{5401}{6531840}h^3 f_{n+\frac{1}{9}} + \frac{5983}{933120}h^3 f_{n+\frac{1}{3}} + \frac{345419}{6531840}h^3 f_{n+\frac{5}{9}} + \frac{227071}{2177280}h^3 f_{n+\frac{7}{9}} + \frac{1040303}{6531840}h^3 f_{n+1} \quad (12)$$

$$hy'_{n+\frac{5}{9}} = 18y_n - \frac{63}{2}y_{n+\frac{1}{9}} + \frac{27}{2}y_{n+\frac{1}{3}} - \frac{17}{241920}h^3 f_n + \frac{361}{1451520}h^3 f_{n+\frac{1}{9}} - \frac{803}{483840}h^3 f_{n+\frac{1}{3}} + \frac{7759}{4354560}h^3 f_{n+\frac{5}{9}} + \frac{21841}{483840}h^3 f_{n+\frac{7}{9}} + \frac{43019}{483840}h^3 f_{n+1} \quad (13)$$

$$hy'_{n+\frac{7}{9}} = 30y_n - \frac{99}{2}y_{n+\frac{1}{9}} + \frac{39}{2}y_{n+\frac{1}{3}} + \frac{31}{2540160}h^3 f_n - \frac{3251}{45722880}h^3 f_{n+\frac{1}{9}} + \frac{479}{1016064}h^3 f_{n+\frac{1}{3}} + \frac{617}{933120}h^3 f_{n+\frac{5}{9}} + \frac{264703}{45722880}h^3 f_{n+\frac{7}{9}} + \frac{54937}{1016064}h^3 f_{n+1} \quad (14)$$

$$hy'_{n+1} = 42y_n - \frac{135}{2}y_{n+\frac{1}{9}} + \frac{51}{2}y_{n+\frac{1}{3}} - \frac{43}{39191040}h^3 f_n + \frac{773}{78382080}h^3 f_{n+\frac{1}{9}}$$
$$-\frac{1025}{15676416}h^3 f_{n+\frac{1}{3}} - \frac{3077}{26127360}h^3 f_{n+\frac{5}{9}} - \frac{5893}{15676416}h^3 f_{n+\frac{7}{9}}$$
$$+\frac{238813}{78382080}h^3 f_{n+1} \qquad (15)$$

Differentiating (5) twice, we have

$$h^2 y''(x) = \lambda''_0 y_n + \lambda''_{\frac{1}{9}} y_{n+\frac{1}{9}} + \lambda''_{\frac{1}{3}} y_{n+\frac{1}{3}} + h^3\left[\phi''_0 f_n + \phi''_{\frac{1}{9}} f_{n+\frac{1}{9}} + \phi''_{\frac{1}{3}} f_{n+\frac{1}{3}} + \phi''_{\frac{5}{9}} f_{n+\frac{5}{9}} + \phi''_{\frac{7}{9}} f_{n+\frac{7}{9}} + \phi''_1 f_{n+1}\right] (16)$$

The second derivatives of the parameter $\lambda_j's, \phi_j's$ are written as the following functions of t.

$$\lambda''_0(t) = 54$$
$$\lambda''_{\frac{1}{9}}(t) = -81$$
$$\lambda''_{\frac{1}{3}}(t) = 27$$

$$\phi''_0(t) = -\frac{10525}{285768}h^3 + t\,h^3 - \frac{563}{70}t^2h^3 + \frac{190}{7}t^3h^3 - \frac{621}{14}t^4h^3 + \frac{243}{7}t^5h^3 - \frac{729}{70}t^6h^3$$

$$\phi''_{\frac{1}{9}}(t) = -\frac{34579}{322560}h^3 + \frac{2835}{256}t^2h^3 - \frac{837}{16}t^3h^3 + \frac{25029}{256}t^4h^3 - \frac{6561}{80}t^5h^3$$
$$+\frac{6561}{256}t^6h^3$$

$$\phi''_{\frac{1}{3}}(t) = -\frac{7093}{2177280}h^3 - \frac{315}{64}t^2h^3 + \frac{687}{16}t^3h^3 - \frac{3321}{32}t^4h^3 + \frac{8019}{80}t^5h^3 - \frac{2187}{64}t^6h^3$$

$$\phi''_{\frac{5}{9}}(t) = -\frac{229}{161280}h^3 + \frac{1701}{640}t^2h^3 - \frac{405}{16}t^3h^3 + \frac{9477}{128}t^4h^3 - \frac{6561}{80}t^5h^3$$
$$+\frac{19683}{640}t^6h^3$$

$$\phi''_{\frac{7}{9}} = (t)\frac{229}{338688}h^3 - \frac{405}{448}t^2h^3 + \frac{999}{112}t^3h^3 - \frac{3159}{112}t^4h^3 + \frac{19683}{560}t^5h^3 - \frac{6561}{448}t^6h^3$$

$$\phi''_1(t) = -\frac{2987}{26127360}h^3 + \frac{35}{256}t^2h^3 - \frac{11}{8}t^3h^3 + \frac{1161}{256}t^4h^3 - \frac{243}{40}t^5h^3 + \frac{729}{256}t^6h^3$$

The second derivatives $\lambda_j's, \phi_j's$ at $t = 0, \frac{1}{9}, \frac{1}{3}, \frac{5}{9}, \frac{5}{9}, 1$ the values obtained are then substituted in to (16) to obtain the following implicit hybrid block scheme.

$$h^2y''_n = 54y_n - 81y_{n+\frac{1}{9}} + 27y_{n+\frac{1}{3}} - \frac{10525}{285768}hh^3 f_n + \frac{8611}{1428840}h^3 f_{n+\frac{1}{9}}$$
$$-\frac{15581}{1428840}h^3 f_{n+\frac{1}{3}} - \frac{1145}{285768}h^3 f_{n+\frac{5}{9}} - \frac{15581}{1428840}h^3 f_{n+\frac{7}{9}}$$
$$+\frac{8611}{1428840}h^3 f_{n+1} \qquad (17)$$

$$h^2y''_{n+\frac{1}{9}}=54y_n-81y_{n+\frac{1}{9}}+27y_{n+\frac{1}{3}}-\frac{34579}{322560}h^3f_n-\frac{83263}{2903040}h^3f_{n+\frac{1}{9}}+\frac{29177}{322560}h^3f_{n+\frac{1}{3}}+\frac{196289}{2903040}h^3f_{n+\frac{5}{9}}+\frac{51085}{580608}h^3f_{n+\frac{7}{9}}+\frac{13049}{322560}h^3f_{n+1} \quad (18)$$

$$h^2y''_{n+\frac{1}{3}}=54y_n-81y_{n+\frac{1}{9}}+27y_{n+\frac{1}{3}}-\frac{7093}{2177280}h^3f_n-\frac{42037}{2177280}h^3f_{n+\frac{1}{9}}+\frac{54223}{435456}h^3f_{n+\frac{1}{3}}+\frac{552907}{2177280}h^3f_{n+\frac{5}{9}}+\frac{453899}{2177280}h^3f_{n+\frac{7}{9}}+\frac{646091}{2177280}h^3f_{n+1} \quad (19)$$

$$h^2y''_{n+\frac{5}{9}}=54y_n-81y_{n+\frac{1}{9}}+27y_{n+\frac{1}{3}}-\frac{229}{161280}h^3f_n+\frac{641}{96768}h^3f_{n+\frac{1}{9}}-\frac{4009}{161280}h^3f_{n+\frac{1}{3}}+\frac{46213}{483840}h^3f_{n+\frac{5}{9}}+\frac{24833}{96768}h^3f_{n+\frac{7}{9}}+\frac{20183}{161280}h^3f_{n+1} \quad (20)$$

$$h^2y''_{n+\frac{7}{9}}=54y_n-81y_{n+\frac{1}{9}}+27y_{n+\frac{1}{3}}+\frac{229}{338688}h^3f_n-\frac{10061}{5080320}h^3f_{n+\frac{1}{9}}+\frac{11729}{1693440}h^3f_{n+\frac{1}{3}}-\frac{6313}{1016064}h^3f_{n+\frac{5}{9}}+\frac{461683}{5080320}h^3f_{n+\frac{7}{9}}+\frac{531857}{1693440}h^3f_{n+1} \quad (21)$$

$$h^2y''_{n+1}=54y_n-81y_{n+\frac{1}{9}}+27y_{n+\frac{1}{3}}-\frac{2987}{26127360}h^3f_n+\frac{1469}{5225472}h^3f_{n+\frac{1}{9}}-\frac{24911}{26127360}h^3f_{n+\frac{1}{3}}+\frac{14513}{26127360}h^3f_{n+\frac{5}{9}}-\frac{89423}{26127360}h^3f_{n+\frac{7}{9}}+\frac{1813681}{26127360}h^3f_{n+1} \quad (22)$$

Equations (6) - (8), equations (10) - (14) and equations (16) - (22) are then put in matrix form to produce:

$$ST_Q=ST_{Q-1}+RF_{Q-1}+UF_Q \quad (23)$$

Where,

$$T_Q=\left[y_{n+\frac{1}{9}},y_{n+\frac{1}{3}},y_{n+\frac{5}{9}},y_{n+\frac{7}{9}},y_{n+1},y'_{n+\frac{1}{9}},y'_{n+\frac{1}{3}},y'_{n+\frac{5}{9}},y'_{n+\frac{7}{9}},y'_{n+1},y''_{n+\frac{1}{9}},y''_{n+\frac{1}{3}},y''_{n+\frac{5}{9}},y''_{n+\frac{7}{9}},y''_{n+1}\right]^T$$

$$T_{Q-1}=\left[y_n,y'_n,y''_n\right]^T,\ F_{Q-1}=\left[f_n\right]^T$$

$$F_Q=\left[f_{n+\frac{1}{9}},f_{n+\frac{1}{3}},f_{n+\frac{5}{9}},f_{n+\frac{7}{9}},f_{n+1}\right]^T$$

The block matrices in equation (24) is then resolved by multiplying by S^{-1} to gives the following discrete scheme

$$y_{n+\frac{1}{9}} = y_n + \frac{1}{9} h y'_n + \frac{1}{162} h^2 y''_n + \frac{33989}{231472080} h^3 f_n + \frac{1067}{10450944} h^3 f_{n+\frac{1}{9}} - \frac{11351}{352719360} h^3 f_{n+\frac{1}{3}} + \frac{433}{26127360} h^3 f_{n+\frac{5}{9}} - \frac{1517}{274337280} h^3 f_{n+\frac{7}{9}} + \frac{3503}{4232632320} h^3 f_{n+1} \quad (25)$$

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3} h y'_n + \frac{1}{18} h^2 y''_n + \frac{191}{105840} h^3 f_n + \frac{2759}{645120} h^3 f_{n+\frac{1}{9}} + \frac{1}{41472} h^3 f_{n+\frac{1}{3}} + \frac{11}{107520} h^3 f_{n+\frac{5}{9}} - \frac{47}{1128960} h^3 f_{n+\frac{7}{9}} + \frac{13}{1935360} h^3 f_{n+1} \quad (26)$$

$$y_{n+\frac{5}{9}} = y_n + \frac{5}{9} h y'_n + \frac{25}{162} h^2 y''_n + \frac{4625}{944784} h^3 f_n + \frac{550625}{31352832} h^3 f_{n+\frac{1}{9}} + \frac{443125}{70543872} h^3 f_{n+\frac{1}{3}} - \frac{3625}{15676416} h^3 f_{n+\frac{5}{9}} + \frac{625}{7838208} h^3 f_{n+\frac{7}{9}} - \frac{8125}{846526464} h^3 f_{n+1} \quad (27)$$

$$y_{n+\frac{7}{9}} = y_n + \frac{7}{9} h y'_n + \frac{49}{162} h^2 y''_n + \frac{44933}{4723920} h^3 f_n + \frac{890771}{22394880} h^3 f_{n+\frac{1}{9}} + \frac{1226911}{50388480} h^3 f_{n+\frac{1}{3}} + \frac{16807}{3732480} h^3 f_{n+\frac{5}{9}} + \frac{343}{1119744} h^3 f_{n+\frac{7}{9}} - \frac{16807}{604661760} h^3 f_{n+1} \quad (28)$$

$$y_{n+1} = y_n + h y'_n + \frac{1}{2} h^2 y''_n + \frac{61}{3920} h^3 f_n + \frac{729}{10240} h^3 f_{n+\frac{1}{9}} + \frac{963}{17920} h^3 f_{n+\frac{1}{3}} + \frac{729}{35840} h^3 f_{n+\frac{5}{9}} + \frac{729}{125440} h^3 f_{n+\frac{7}{9}} + \frac{1}{43008} h^3 f_{n+1} \quad (29)$$

$$y'_{n+\frac{1}{9}} = y'_n + \frac{1}{9} h y''_n + \frac{5449}{1607445} h^2 f_n + \frac{4411}{1306368} h^2 f_{n+\frac{1}{9}} - \frac{4583}{4898880} h^2 f_{n+\frac{1}{3}} + \frac{173}{362880} h^2 f_{n+\frac{5}{9}} - \frac{1811}{11430720} h^2 f_{n+\frac{7}{9}} + \frac{1391}{58786560} h^2 f_{n+1} \quad (30)$$

$$y'_{n+\frac{1}{3}} = y'_n + \frac{1}{3} h y''_n + \frac{214}{19845} h^2 f_n + \frac{1039}{26880} h^2 f_{n+\frac{1}{9}} + \frac{85}{12096} h^2 f_{n+\frac{1}{3}} - \frac{1}{896} h^2 f_{n+\frac{5}{9}} + \frac{11}{47040} h^2 f_{n+\frac{7}{9}} - \frac{19}{725760} h^2 f_{n+1} \quad (31)$$

$$y'_{n+\frac{5}{9}} = y'_n + \frac{5}{9} h y''_n + \frac{5575}{321489} h^2 f_n + \frac{104375}{1306368} h^2 f_{n+\frac{1}{9}} + \frac{53125}{979776} h^2 f_{n+\frac{1}{3}} + \frac{575}{217728} h^2 f_{n+\frac{5}{9}} + \frac{625}{2286144} h^2 f_{n+\frac{7}{9}} - \frac{625}{11757312} h^2 f_{n+1} \quad (32)$$

$$y'_{n+\frac{7}{9}} = y'_n + \frac{7}{9} h y''_n + \frac{784}{32805} h^2 f_n + \frac{112847}{933120} h^2 f_{n+\frac{1}{9}} + \frac{74431}{699840} h^2 f_{n+\frac{1}{3}} + \frac{2401}{51840} h^2 f_{n+\frac{5}{9}} + \frac{245}{46656} h^2 f_{n+\frac{7}{9}} - \frac{2401}{8398080} h^2 f_{n+1} \quad (33)$$

$$y'_{n+1} = y'_n + h y''_n + \frac{23}{735} h^2 f_n + \frac{1431}{8960} h^2 f_{n+\frac{1}{9}} + \frac{363}{2240} h^2 f_{n+\frac{1}{3}} + \frac{81}{896} h^2 f_{n+\frac{5}{9}} + \frac{837}{15680} h^2 f_{n+\frac{7}{9}} + \frac{17}{5376} h^2 f_{n+1} \quad (34)$$

$$y''_{n+\frac{1}{9}} = y''_n + \frac{3}{70} h f_n + \frac{8141}{103680} h f_{n+\frac{1}{9}} - \frac{13}{810} h f_{n+\frac{1}{3}} + \frac{139}{17280} h f_{n+\frac{5}{9}} - \frac{241}{90720} h f_{n+\frac{7}{9}} + \frac{41}{103680} h f_{n+1} \quad (35)$$

$$y''_{n+\frac{1}{3}} = y''_n + \frac{7}{270} h f_n + \frac{253}{1280} h f_{n+\frac{1}{9}} + \frac{23}{180} h f_{n+\frac{1}{3}} - \frac{3}{128} h f_{n+\frac{5}{9}} + \frac{1}{160} h f_{n+\frac{7}{9}} - \frac{29}{34560} h f_{n+1} \quad (36)$$

$$y''_{n+\frac{5}{9}} = y''_n + \frac{335}{10206} h f_n + \frac{3625}{20736} h f_{n+\frac{1}{9}} + \frac{125}{486} h f_{n+\frac{1}{3}} + \frac{335}{3456} h f_{n+\frac{5}{9}} - \frac{125}{18144} h f_{n+\frac{7}{9}} + \frac{125}{186624} h f_{n+1} \quad (37)$$

$$y''_{n+\frac{7}{9}} = y''_n + \frac{7}{270} h f_n + \frac{20237}{103680} h f_{n+\frac{1}{9}} + \frac{343}{1620} h f_{n+\frac{1}{3}} + \frac{4459}{17280} h f_{n+\frac{5}{9}} + \frac{1169}{12960} h f_{n+\frac{7}{9}} - \frac{343}{103680} h f_{n+1} \quad (38)$$

$$y''_{n+1} = y''_n + \frac{3}{70} h f_n + \frac{189}{1280} h f_{n+\frac{1}{9}} + \frac{3}{10} h f_{n+\frac{1}{3}} + \frac{81}{640} h f_{n+\frac{5}{9}} + \frac{351}{1120} h f_{n+\frac{7}{9}} + \frac{89}{1280} h f_{n+1} \quad (39)$$

2.1 Analysis of the Method

In this paper, the main properties of the one-step four-point hybrid block method for solving third order initial value problems are presented. The properties include the order and error constant, zero stability, linear stability, stability polynomial, consistency and convergence of the method.

Consider the linear operator L associated with the implicit hybrid block method (25) – (39) defined as

$$L[y(x_n : h)] = \sum_j \left[\alpha_j y(x_n + jh) - h^3 \beta_j y'''(x_n + jh)\right] \qquad (40)$$

Where $y(x_n)$ is an arbitrary test function that is continuous and differentiable in the interval [a, b]. Obtaining the Taylor series expansions of $y(x_n + jh)$ and $y'''(x_n + jh)$ about x_n and collecting the coefficient of h^p gives;

$$L[y(x_n : h)] = c_0 y(x_n) + c_1 hy'(x_n) + c_2 h^2 y''(x_n) + \ldots.. + c_p h^p y^{(p)}(x_n) + \ldots\ldots \qquad (41)$$

Where $c_j{}'s$ for j = 0, 1, 2, 3….

2.2 Orders and Error Constants

From (35), if it is obtained that:

$$c_0 = c_1 = c_2 = \ldots = c_{p+2} = 0, c_{p+3} \neq 0$$

Then the hybrid block method (25) – (39) is of order 6 and the error constants are

$$\begin{pmatrix} -\frac{91319}{1405871470483200}, -\frac{401}{642831033600}, \frac{7625}{56234858819328}, \frac{218491}{200838781497600}, \frac{1}{293932800}, \\ -\frac{1067}{578547930240}, \frac{1}{3061100160}, \frac{3625}{1041386274432}, \frac{2401}{247949112960}, -\frac{1}{264539520}, \\ -\frac{4411}{144636982560}, \frac{17}{357128352}, -\frac{575}{28927396512}, \frac{343}{4132485216}, -\frac{17}{66134880} \end{pmatrix}^T$$

2.3 Zero Stability of the Block Method

The new block method is zero stable if the first characteristic polynomial

$$\rho(w) = \det\left[\sum_{j=0}^{k} A^{(i)} w^{k-i}\right] = 0 \qquad (42)$$

and satisfies the equation, $|w_j| \leq 1$, the multiplicity must not exceed the order of differential equation. Omole and Ukpebor, [10].

$$B(w) = w\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 1&0&0&0&0&0&\frac{1}{9}&0&0&0&\frac{1}{162}&0&0&0&0 \\ 1&0&0&0&0&0&\frac{1}{3}&0&0&0&\frac{1}{18}&0&0&0&0 \\ 1&0&0&0&0&0&\frac{5}{9}&0&0&0&\frac{25}{162}&0&0&0&0 \\ 1&0&0&0&0&0&\frac{7}{9}&0&0&0&\frac{49}{162}&0&0&0&0 \\ 1&0&0&0&0&0&1&0&0&0&\frac{1}{2}&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&\frac{1}{9}&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&\frac{1}{3}&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&\frac{5}{9}&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&\frac{7}{9}&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} = w^{12}(w-1)^3$$

$$\rho(w) = w^{12}(w-1)^3 = 0, \quad w = ,0,0,0,0,0,0.0,0,0,0,0,0,1,1,1$$

Therefore, $|w_i| = |0,0,0,0,0,0.0,0,0,0,0,0,1,1,1| \leq 1$, the method is zero-stable

2.4 Linear Stability

The linear stability of the method is given by:

$$-(A-B)^{-1}(C-zD)$$

Where,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{8141}{103680} & -\frac{13}{810} & \frac{139}{17280} & -\frac{241}{90720} & \frac{41}{103680} \\ \frac{253}{1280} & \frac{23}{180} & -\frac{3}{128} & \frac{1}{160} & -\frac{29}{34560} \\ \frac{3625}{20736} & \frac{125}{486} & \frac{335}{3456} & -\frac{125}{18144} & \frac{125}{186624} \\ \frac{20237}{103680} & \frac{343}{1620} & \frac{4459}{17280} & \frac{1169}{12960} & -\frac{343}{103680} \\ \frac{189}{1280} & \frac{3}{10} & \frac{81}{640} & \frac{351}{1120} & \frac{89}{1280} \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$, D = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3}{70} \\ 0 & 0 & 0 & 0 & \frac{7}{270} \\ 0 & 0 & 0 & 0 & \frac{335}{10206} \\ 0 & 0 & 0 & 0 & \frac{7}{270} \\ 0 & 0 & 0 & 0 & \frac{3}{70} \end{bmatrix}$$

The eigenvalues of the block method is

$$\left\{0, \frac{-4480z^5 + 60128z^4 + 365400z^3 + 6207840z^2 + 24873480z + 55112400}{1225z^5 - 39410z^4 + 598500z^3 - 5216400z^2 + 25515000z - 55112400}\right\}$$

2.5 Stability Polynomial

The stability polynomial of a linear multi-step the method is given by:
Aw-C-E-h D-h B w

Where;

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{8141}{103680} & -\frac{13}{810} & \frac{139}{17280} & -\frac{241}{90720} & \frac{41}{103680} \\ \frac{253}{1280} & \frac{23}{180} & -\frac{3}{128} & \frac{1}{160} & -\frac{29}{34560} \\ \frac{3625}{20736} & \frac{125}{486} & \frac{335}{3456} & -\frac{125}{18144} & \frac{125}{186624} \\ \frac{20237}{103680} & \frac{343}{1620} & \frac{4459}{17280} & \frac{1169}{12960} & -\frac{343}{103680} \\ \frac{189}{1280} & \frac{3}{10} & \frac{81}{640} & \frac{351}{1120} & \frac{89}{1280} \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3}{70} \\ 0 & 0 & 0 & 0 & \frac{7}{270} \\ 0 & 0 & 0 & 0 & \frac{335}{10206} \\ 0 & 0 & 0 & 0 & \frac{7}{270} \\ 0 & 0 & 0 & 0 & \frac{3}{70} \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 0 & \frac{7}{9} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|Aw - C - E - hD - hBw| = \left(-\frac{7}{314928}w^5 - \frac{8}{98415}w^4\right)h^5 + \left(\frac{563}{787320}w^5 - \frac{2819}{1574640}w^4\right) h^4 + \left(-\frac{95}{8748}w^5 - \frac{349}{17496}w^4\right)h^3 + \left(\frac{23}{243}w^5 - \frac{157}{972}w^4\right)h^2 + \left(-\frac{25}{54}w^5 - \frac{31}{54}w^4\right)h + (w^5 - 2w^4$$

The region of absolute stability of the method is plotted and shown in figure1.

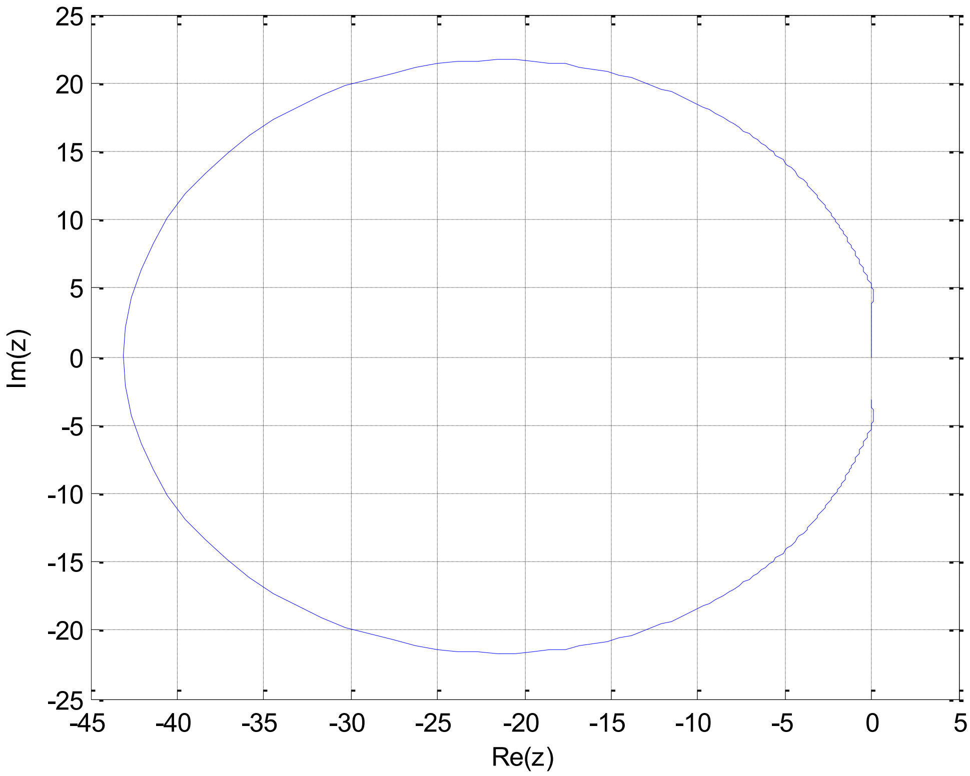

Figure1. The region of absolute stability of the method

2.6 Consistency of the Method

The linear multistep method is said to be consistent if it has order $p \geq 1$, [11]. The derived hybrid method is consistent since the order is 6 greater than one.

2.7 Convergence of the method

The method is said to be convergent if and only if it is consistent and zero stable, [11]. Since the proposed method satisfies the two conditions, then the method converges

2.8 Numerical implementation of the scheme

We shall evaluate the performance of the block method on some problems which appear in literature and compare the results with our methods. The numerical results are obtained using Maple software.

Example1. Consider the initial value problem below

$$y'''-y''+y'-y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01$$

The Exact solution is $y(x) = \cos x$

New method is compared in terms of absolute error in first block method with S=52 of order 6 and error in second block method with S=94 in Kuboye et al, [12].

Example2. Consider a highly stiff problem

$$y'''+5y''+7y'+3y = 0 \text{ , } y(0) = 1 \text{ , } y'(0) = 0 \text{ , } y''(0) = -1 \text{ ,}$$

Exact Solution: $y(x)=e^{-x}+xe^{-x}, \quad h=\frac{1}{10}$

New method is compared in terms of absolute error with error in Tumba *et al.,* [13].

Table 01. Showing the comparison of absolute error in our method with kuboye et al, [12] for example one

X	Exact solution	Computed solution	Error in our method	Error in kuboye S=52 [11]	Error in kuboye S=94 [12]
0.01	0.9999500004166652777	0.9999500004166652777	0.0000000	1.1102230e-16	1.000000e+00
0.02	0.9998000066665777841	0.9998000066665777842	1.0000e-20	5.5511151e-16	5.5511151e-16
0.03	0.9995500337489875 1627	0.9995500337489875 1628	1.0000e-20	8.6597396e-15	8.7707619e-15
0.04	0.9992001066609779 4031	0.9992001066609779 4033	2.0000e-20	6.4837025e-14	6.4614980e-14
0.05	0.9987502603949662 4656	0.9987502603949662 4657	1.0000e-20	2.6301183e-14	2.6290081e-14

Table 02. Showing the comparison of absolute error in our method with Tumba *et al.* [13] for problem two

x	Exact solution	Computed solution	Error in our method	Error in Tumba *et al.* [13].
0.1	0.99532115983955553048	0.99532115983955550649	2.399e-17	1.0434e-14
0.2	0.98247690369357823040	0.98247690369357907217	8.4177e-16	9.8731e-14
0.3	0.96306368688623322589	0.96306368688623690353	3.67764e-15	3.1317e-13
0.4	0.93844806444989502104	0.93844806444990388985	8.86881e-15	6.6668e-13
0.5	0.90979598956895013540	0.90979598956896650456	1.636916e-14	1.1507e-12
0.6	0.87809861775044229221	0.87809861775046817516	2.588295e-14	1.7445e-12
0.7	0.84419501644539617499	0.84419501644543316253	3.698754e-14	2.4220e-12
0.8	0.80879213541099886457	0.80879213541104807872	4.921415e-14	3.1554e-12
0.9	0.77248235350713831257	0.77248235350720041186	6.209929e-14	3.9178e-12
1.0	0.73575888234288464320	0.73575888234295985875	7.521555e-14	4.6852e-12

2.9 Conclusion

In this paper, one-step block method with four hybrid points for the numerical solution of third order initial value problems is derived and implemented. The method was derived through interpolation of the assumed power series solution and four off-grid points. The third derivative of the assumed solution was collocated at all points in the interval of consideration. The properties of the method including order and error constant, consistency, zero stability, Linear Stability, Stability Polynomial and convergence were discussed. Numerical results were presented in Table 01- 02, shows that table 01 has been compare with Kuboye et al, [12] of the same order with improved performance in our method. Table 02 has been compared with Tumba *et al.* [13] of the same order with improved performance in our method.

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