OPEN ACCESS
Global Journal of Research in Engineering & Computer Sciences
ISSN: 2583-2727 (Online)
Volume 05| Issue 04 | July-Aug. | 2025
Journal homepage: https://gjrpublication.com/gjrecs/
Research Article

A Numerical Method for Solving the Initial Value Problems of Fractional Order Volterra Integral-Differential Equations

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DOI: 10.5281/zenodo.17009302 **Submission Date:** 18 July 2025 | **Published Date:** 30 Aug. 2025

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Abstract
In this paper, a numerical approach is developed for solving the initial value problem of linear Volterra integro-differential equations. The approximate solution is substituted into the model equation and then collocated using shifted Chebyshev polynomials and standard collocation points to obtain a system of linear algebraic equations, which is then solved by Newton-Raphson's method. Several numerical examples were solved to demonstrate the accuracy, reliability, and efficiency of the method.

Keywords: *Integro-differential equations, Standard collocation points, Shifted Chebyshev polynomial, Newton-Rapson's method.*

1. Introduction
Many disciplines including physics, mathematics, engineering, and chemistry, use fractional differential and integral equations. Real-world problems that can be quantitatively expressed as functional equations include ordinary differential equations and partial differential equations. Physical phenomena are modeled using integral-differential equations in science and engineering. They can be found in many mathematical representations of physical phenomena (IDEs). Kinetic equations defining the kinetic theory of rarefied gases, plasma, coagulation, and radiation transport are some of the challenges [1].

Here are a few examples of numerical solutions to fractional differential equations that have been developed in the literature: Perturbed collocation approach [2], Adomian decomposition method [3-5], collocation method [6-9], hybrid linear multi-step method [10-11], differential transform method [12], pseudo-spectral method [13], Bernstein polynomials method [14-15], [16]; a numerical technique based on the Boubaker polynomial They selected to use a Boubakar polynomial as their operational matrix for fractional integration.

In this paper, we address the numerical solution of the fractional order Volterra integro-differential equation of the form:

$$D^{\alpha} y(x) = f(x) + p(x)y(x) + \lambda\int_0^x K(x,t)F(y(t))dt, \quad 0 \le x \le 1 \tag{1}$$

Subject to the following initial condition:

$$y^{m-1}(0) = y_{m-1}, \quad m-1 < \alpha \le m \tag{2}$$

where $m > 0$; $y(t)$ and $K(x;t)$ are analytic functions. $y(x)$ is the unknown to be determined, the upper limit x is a variable while $D^{\alpha} y(x) = {}_0^C D_t^{\alpha} y(t)$ and they satisfies Lipschitz conditions.

2. Preliminaries

In this section, we consider several basic definitions and properties of fractional calculus theory and shifted Chebyshev polynomial which is used as basis function.

2.1 Caputo derivatives [18]

The left and right Caputo derivative with order α > 0 of the given function $f(t),\ t \in (a,b)$ are defined as:

$$ {}_{\alpha}^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} f^{(m)}(s)ds \tag{3}$$

and

$$ {}_{\alpha}^{C}D^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} f^{(m)}(s)ds \tag{4}$$

respectively, where m is a positive integer satisfying $m-1 \le \alpha < m.$

2.2 Standard collocation point [6]

The method is used to determine the desired collocation points within an interval, say $[\vartheta, \sigma]$, and is given by

$$ x_p = \vartheta + \frac{(\sigma - \vartheta)}{N} p, \quad p = 1, 2, \ldots, N \tag{5}$$

2.3 Shifted Chebyshev polynomial [22]

The Chebyshev polynomials are defined on the interval [-1,1]; with the recurrence formula

$$ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ n = 1, 2, \ldots \tag{6}$$

where $T_0(x) = 1$; $T_1(x) = x$: The analytic form of degree n is given by

$$ T_n(x) = \frac{n}{2} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r (n-r-1)}{r!(n-2r)!} (2x)^{n-2r} \tag{7}$$

In order to apply the polynomial in the interval [0,1]; shifted Chebyshev polynomial $T_n^*(x)$ is defined as:

$$ T_n^*(x) = T_n(2x-1) \tag{8}$$

with the following recurrence formula

$$ T_{n+1}^*(x) = 2(2x-1)T_n^*(x) - T_{n-1}^*(x), \ n = 1, 2, \ldots \tag{9}$$

where $T_0^*(x) = 1,\ T_1^*(x) = 2x-1.$ The analytical form of the shifted polynomial is given as:

$$ T_n^*(x) = \sum_{r=0}^{n} (-1) 2^{2n-2r} \frac{n(2n-r-1)}{r!(2n-2r)!} x^{n-r} \tag{10}$$

2.4 Properties of Caputo fractional integrals and derivatives. [21]

If $\alpha \in C(\mathrm{Re}(\alpha) > 0)$ and $\beta \in C(\mathrm{Re}(\alpha) > 0)$; then:

(i) $$ \left(I^{\alpha} (t-\alpha)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-\alpha)^{\beta+\alpha-1} \tag{11}$$

(ii) $$ \left(D^{\alpha} (t-\alpha)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-\alpha)^{\beta-\alpha-1} \tag{12}$$

2.5 Properties of Caputo fractional integrals and derivatives. [24]

The Caputo fractional operator I^{α} of order α has the following properties:

(i) $$I^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma} \quad (13)$$

(ii) $$I^{\alpha_1}I^{\alpha_2}f(x) = I^{\alpha_1+\alpha_2}f(x),\ \alpha_1,\alpha_2 > 0 \quad (14)$$

(iii) $$D^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha-\gamma+1)}x^{\alpha-\gamma} \quad (15)$$

(iv) $D^{\alpha}C = 0$, C is a constant. (16)

(vi) $$D^{\alpha}I^{\alpha}f(x) = f(x),\ x > 0 \quad (17)$$

3. Materials and Methods

Lemma 1.

Let $y(x)$ be a continuous function, then

$$_0I_t^{\alpha}\left({}_0^CD_t^{\alpha}y(x)\right) = y(t) - \sum_{k=0}^{m-1}\frac{x^k}{k!}y(k)(0) \quad (18)$$

where $m-1 < \alpha \leq m$.

Lemma 2.

Let $u(t)$ be an integrable function, then

$$_0I_x^{n}(u(x)) = u(x) = \frac{1}{\Gamma}\int_0^x (x-t)^{\alpha-1}u(t)dt \quad (19)$$

3.1 Transforming the model equation into an integral equation.

Lemma 3.

Let $y(x)$ be the solution to (1) and (2), then

$$y(x) = H(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)y(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_0^t K(t,s)F(y(s))ds\right]dt \quad (20)$$

where

$$H(x) = \sum_{k=0}^{m-1}\frac{x^k}{k!}y^{(k)}(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\ m-1 < \alpha \leq m$$

Lemma 4.

Let $T : C([0, 1, \mathbb{R}]) \rightarrow C([0, 1, \mathbb{R}])$, be fixed by (19), then

$$(Ty)(x) = H(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}p(t)y(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_0^t K(t,s)F(y(s))ds\right]dt$$
$$= y(x) \quad (21)$$

3.2 Transforming the integral equations into system of algebraic equations.

***Theorem* 1.**

Let the solution to (1) and (2) be approximated by the shifted Chebyshev polynomial

$$y(x) = \mathrm{T}(X)\mathrm{A} \quad (22)$$

where

$$T(X) = \left[T_0^*\ T_1^*(x)\cdots T_N^*\right]$$

where $T_N^*(x)$ is the shifted Chebyshev polynomial defined by (8) and

$$A = [a_0\, a_1 \ldots a_N]^T \qquad (23)$$

are the unknown coefficients to be determined.

Substituting (22) into (20) and collocating at x_i; $i = 0(1)N$ gives

$$0 = T(x_i) - H(x) + \frac{1}{\Gamma(\alpha)}\left(\int_0^x (x-t)^{\alpha-1} p(t)T(t)dt\right)A - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_0^t K(t,s)F(T(s)A)ds\right]dt \qquad (24)$$

Equation (23) is $(N+1) \times (N+1)$ linear equation which is then solved using Newton Raphson's method.

***Lemma* 5.**

Let $y(x)$ be approximated by (22); then

$$y(x) = x^n \beta A, \qquad (25)$$

where

$$\beta = \begin{bmatrix} m(0,0) & 0 & 0 & \cdots & 0 \\ m(1,1) & m(1,0) & 0 & \cdots & 0 \\ m(2.2) & m(2,1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m(N,N) & m(N,N-1) & m(N,N-2) & \cdots & m(N,0) \end{bmatrix}$$

and from (10), we have

$$m(n,r) = \frac{(-1)2^{2n-2r} n\Gamma(2n-r)}{\Gamma(r+1)\Gamma(2n-2r+1)}, \quad n = 0(1)N$$

***Lemma* 6.**

Let $y(x)$ be approximated by (21); and

$$U(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_0^t K(t,s)F(y(s))ds\right]dt \qquad (26)$$

writing $K(t;s)$ as $t^i s^j$ and $F(y(s))$ as $y^m(s)$; then

$$U(x;n) = \frac{\Gamma(nm+j+i+2)}{\Gamma(\alpha+nm+j+i+2)} x^{\alpha+nm+j+i+2} \beta A \qquad (27)$$

***Lemma* 7.**

Let $y \in ([0, 1], \mathbb{R})$ and $p(t) \in ([0, 1])$ be defined by $p(t) = t^j$, if

$$V(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} p(t)y(t)dt \qquad (28)$$

then

$$V(x;n) = \frac{\Gamma(j+n+1)}{\Gamma(\alpha+j+n+1)} x^{\alpha+j+n+1} \beta A \qquad (29)$$

***Theorem* 2.**

Let $y(x)$ the solution of (1) and (2) be approximated by (20); then it is equivalent to

$$\mathbf{F}(x;n) = 0;\ n = 0(1)N;\ N \in \mathbb{Z}^+ \qquad (30)$$

where

$$F(x;n)=\left(x_i^n-\frac{\Gamma(j+n+1)}{\Gamma(\alpha+j+n+1)}\right)\beta A$$
$$-\frac{\Gamma(nm+j+2)}{\Gamma(\alpha+nm+j+i+2)(nm+j+i)}x_i^{\alpha+nm+j+i+1}\beta^m A^m-Q(x_i)$$

and

$$Q(x_i)=\left(\sum_{k=0}^{m-1}\frac{x_i^k}{k!}y^{(k)}+\frac{\Gamma(s+1)}{\Gamma(\alpha+s+1)}x_i^{\alpha+s}\right)$$

4. Convergence of the method
Theorem 3.

Let $y(t)$ and $y_n(t)$ be the exact and the approximate solutions of (1) and (2) respectively, then the solution of method converges if;

$$d(y,y_N)\le\frac{\Gamma(n+1)d(H,H_N)+d(p,p_N)y_N^*}{\Gamma(n+1)-Lp^*-Lk^*}$$

where $y_N^*=\max|y_N|$, $p^*=\max|p(t)|$, $k^*=\max\int_0^x|k(x,t)|dt.$

Proof

Let $H(x)$ and $p(t)$ be expanded by shifted Chebyshev polynomial, then

$$|y(x)-y_N(x)|\le|H(x)-H_N(x)|+\frac{1}{\Gamma(n+1)}\int_0^x(x-t)^{\alpha-1}|p(t)||y(t)-y_N(t)|dt$$
$$+\frac{1}{\Gamma(n+1)}\int_0^x(x-t)^{\alpha-1}|y(t)||p(t)-p_N(t)|dt$$
$$+\frac{1}{\Gamma(n+1)}\int_0^x(x-t)^{\alpha-1}\left[\int_0^t|K(t,s)||y(t)-y_N(t)|ds\right]dt$$

Taking the maximum of both sides, we get

$$d(y,y_N)\le d(H,H_N)+\frac{Lp^*d(y,y_N)}{\Gamma(n+1)}+\frac{d(y,y_N)y_N^*}{\Gamma(n+1)}+\frac{Lk^*d(y,y_N)}{\Gamma(n+1)}$$

Hence,

$$d(y,y_N)=\frac{\Gamma(n+1)d(H,H_N)+d(p,p_N)y_N^*}{\Gamma(n+1)-Lp^*-Lk^*}$$

hence $d(y,y_N)\le 0$, this implies that the method converges.

5. Numerical Illustrations

The numerical examples to test the efficiency of the method are presented in tabular form except where it gives exact results. All computations are done with the aid of program in MATLAB 2015a. $Error=|y_n(x)-y(x)|$. The collocation point considered is

$$x_i=a+\frac{b-a}{N}i,\ [a,b]=[0,1]$$

Example 1 *We consider the following fractional order nonlinear integro-differential equation.* [23]

$$ {}_0^C D^{\alpha} u(t) = \sinh t + \frac{1}{2}\cosh t \sinh t - \frac{t}{2} - \int_0^t u^2(t) ds $$

subject to the initial condition,

$$ u(0) = 0,\ u'(0) = 0 $$

the exact solution is at $\alpha = 2$ is $u(t) = \sinh t$.

Solution

Comparing with (1.1);

$$ f(t) = \sinh t + \frac{1}{2}\cosh t \sinh t - \frac{t}{2}, $$
$$ p(t) = 0,\ K(t,s) = 1,\ \lambda = 1 $$

we have

$$ f(x) = \sum_{n=0}^{N} \frac{x^{2n+1}}{(2n+1)!} + \frac{1}{2}\left(\sum_{n=0}^{N} \sum_{m=0}^{N} \frac{x^{2n+2m+1}}{(2n+1)!(2m)!} \right) - \frac{x}{2} $$

hence

$$ D^{\alpha} u(t) = \sum_{n=0}^{N} \frac{x^{2n+1}}{(2n+1)!} + \frac{1}{2}\left(\sum_{n=0}^{N} \sum_{m=0}^{N} \frac{x^{2n+2m+1}}{(2n+1)!(2m)!} \right) - \frac{t}{2} - \int_0^t u^2(s) ds $$

Writing in integral form, we have

$$ \begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{N} \frac{t^{2n+1}}{(2n+1)!} \int_0^t (s-t)^{\alpha-1} t^{2n-1} dt \\ &+ \frac{1}{2\Gamma(\alpha)} \sum_{n=0}^{N} \sum_{m=0}^{N} \frac{1}{(2n+1)!(2m)!} \int_0^t (s-t)^{\alpha-1} t^{2n+1+2m} dt \\ &- \frac{1}{2\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} t dt - \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \left[\int_0^t u^2(s) ds \right] dt \end{aligned} $$

Using (11)

$$ \begin{aligned} u(t) &= \sum_{n=0}^{N} \frac{\Gamma(2n)}{\Gamma(2n+2)\Gamma(\alpha+2n+2)} t^{\alpha+2n+1} \\ &+ \frac{1}{2} \sum_{n=0}^{N} \sum_{m=0}^{N} \frac{\Gamma(2n+2m+2)}{\Gamma(2n+2)\Gamma(2m+1)\Gamma(\alpha+2n+2m+2)} \\ &- \frac{1}{2\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} \left[\int_0^t u^2(s) ds \right] dt \end{aligned} $$

Using the approximate solution, we have

$$ \begin{aligned} X(t)\beta A &= \sum_{n=0}^{N} \frac{\Gamma(2n)}{\Gamma(2n+2)\Gamma(\alpha+2n+2)} t^{\alpha+2n+1} \\ &+ \frac{1}{2} \sum_{n=0}^{N} \sum_{m=0}^{N} \frac{\Gamma(2n+2m+2)}{\Gamma(2n+2)\Gamma(2m+1)\Gamma(\alpha+2n+2m+2)} \\ &- \frac{1}{2\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-t)^{\alpha-1} \left[\int_0^t (X(s)\beta A)^2 ds \right] dt \end{aligned} $$

Using N = 4 for illustration, we get

$$T(t) = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \frac{1164}{1165} & \frac{71}{2510} & \frac{369}{2510} & \frac{83}{3783} \end{bmatrix}^T$$

$$t_i = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix}$$

$$u(t_i) = \begin{bmatrix} 0 & \frac{25}{768} & \frac{7}{96} & \frac{33}{256} & \frac{5}{24} \end{bmatrix}$$

$$u_4(t) = 0.02194028\, t^4 + 0.147012\, t^3 + 0.007107819\, t^2 + 0.9991413\, t$$

$$u_6(t) = 0.00072879 t^6 + 0.007297872 t^5 + 0.0007341664 t^4 + 0.1663942 t^3 + 0.00004957943 t^2 + 0.9999966 t$$

MATLAB 15a is then used to evaluate.

Table 1: Result for Example 1

T	N = 2	N = 4	N = 6	N = 8	Exact solution
0.2	0.19247694	0.20132377	0.20133601	0.20133600	0.20136000
0.4	0.40625534	0.41076421	0.41075231	0.41075233	0.41075233
0.6	0.64133518	0.63664165	0.63665359	0.63665359	0.63665358
0.8	0.89771646	0.88811893	0.88810597	0.88810599	0.88810598
1.0	1.17539920	1.17520140	1.17520120	1.17520120	1.7520120

Table 2: Error in Result of Example 1

T	$ERR_{N=2}$	$ERR_{N=4}$	$ERR_{N=6}$	$ERR_{N=8}$
0.2	8.859100e-03	1.222900e-05	1.086400e-08	1.517600e-09
0.4	4.49700e-03	1.189400e-05	1.429700e-08	3.206200e-09
0.6	4.681600e-03	1.193500e-05	6.606900e-08	5.285100e-09
0.8	9.610500e-03	1.294500e-05	1.153600e-08	7.901600e-09
1.0	1.980100e-03	2.053600e-07	1.508600e-09	1.122200e-09

Example 2 Consider the following linear fractional Volterra integro-differential equation: [19]

$${}_{O}^{C}D^{0.5}u(t) + u(t) = \frac{2}{s} + \frac{3\sqrt{\pi}}{4}t + t^{\frac{3}{2}} - \frac{2e^{t^{\frac{3}{2}}}}{5} + \int_0^t te^{ts^{2\sqrt{2}}}\,ds$$

with the initial conditions u (0) = 0 and

the exact solution of this equation is $u(t) = t^{\frac{3}{2}}$:

Solution 2

Comparing with (1), we get $\alpha = \frac{1}{2}$, $p(t) = -1$, $K(t,s) = te^{ts^{2\sqrt{2}}}$, $f(t) = \frac{2}{s} + \frac{3\sqrt{\pi}}{4}t + t^{\frac{3}{2}} - \frac{2e^{t^{\frac{3}{2}}}}{5}$.

Following the same procedure as in example 1, we get the integral form as:

$$u(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1}\left(\frac{2}{s} + \frac{3\sqrt{\pi}}{4}t - \frac{2e^{t^{\frac{3}{2}}}}{5}\right)ds + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1}\left[\int_0^s \tau e^{\tau s^{2\sqrt{2}}}\,d\tau\right]ds - \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1}u(t)ds$$

Which is equivalent to:

$$u(t)=\frac{2}{\Gamma\left(\frac{1}{2}\right)}t^{-\frac{1}{2}}+\frac{3\sqrt{\pi}\Gamma(2)}{4\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}}+\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}t^{2}+\frac{\Gamma(2)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}}$$

$$+\frac{2}{5\Gamma\left(\frac{1}{2}\right)}\sum_{i=0}^{N}\frac{1}{\Gamma(i+1)}\left[\frac{\Gamma\left(\frac{3}{2}i+1\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}i+1\right)}t^{\frac{1}{2}+\frac{3}{2}i}\right]+\frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{i=0}^{N}\frac{1}{\Gamma(i+1)}\left[\frac{\Gamma\left(i+1+\frac{5}{2}i+2\right)}{\Gamma\left(\frac{7}{2}+\frac{7}{2}i\right)\left(1+\frac{7}{2}i\right)}t^{\frac{1}{2}+\frac{3}{2}i}\right]$$

Taking $N=2$ for Illustration

$$T(t)=\begin{bmatrix}1 & t & t^2\end{bmatrix},\ t_i=\begin{bmatrix}0 & \frac{1}{2} & 1\end{bmatrix},\ u(t_i)=\begin{bmatrix}0 & 0.5071978886\ 83591 & 1.3738563927\ 9996\end{bmatrix}$$

$$A=\begin{bmatrix}a_0 & a_1 & a_2\end{bmatrix}^T=\begin{bmatrix}\frac{1352}{3375} & \frac{793}{1583} & \frac{198}{1973}\end{bmatrix}^T$$

Evaluating using MATLAB code we get

$$u_2(t)=0.6021285275\ 94955\,t^2+0.3997662396\ 44027\,t-2.7755575615\ 6289\,e-17$$

$$u_3(t)=-0.26678142168088t^3+0.993572760734585t^2$$
$$+0.272952653763939t-6.07153216591882e-17$$

$$u_5(t)=-0.364909557430964t^5+1.18488774141453t^4-1.60802634204018t^3$$
$$+1.6080877375t^2+0.17992264603432t+2.05998412772246e-17$$

Table 3: Result for Example 2

t	N = 3	N = 5	N = 8	N = 10	Exact solution
0.2	0.09219919	0.08922877	0.089384369	0.089430102	0.08944272
0.4	0.25107869	0.25294586	0.252965920	0.252976540	0.25298221
0.6	0.46383300	0.46471757	0.464755230	0.464755710	0.46475800
0.8	0.71765660	0.72556124	0.715553040	0.715541910	0.71554175
1.0	0.99974399	0.99996223	1.000015800	1.000008100	1.00000000

Table 4: Error in Result of Example 2

t	$ERR_{N=3}$	$ERR_{N=5}$	$ERR_{N=8}$	$ERR_{N=10}$
0.2	2.7565e-03	2.1984e-04	5.8350e-05	1.2617e-05
0.4	1.9035e-03	3.6350e-05	1.6295e-05	5.6703e-06
0.6	9.2500e-04	4.0434e-05	2.7740e-06	2.2904e-06
0.8	2.1148e-03	1.9484e-05	1.1283e-05	1.5925e-07
1.0	2.5601e-04	3.7774e-05	1.5812e-05	8.1187e-06

Example 3. Consider the following linear fractional integro-differential equation [20]

$${}_{0}^{C}D_t^{\frac{1}{4}}u(t)=\frac{3\sqrt{\pi}}{4\Gamma\left(\frac{13}{6}\right)}t^{\frac{7}{6}}-\frac{2}{63}t^{\frac{9}{2}}\left(9-7t^2\right)+\int_0^t\left(ts-t^2s^2\right)u(s)dt$$

with the initial condition $u(0)=0$,

and the exact solution at $\alpha=\frac{1}{4}$ *is* $u(t)=t^{\frac{3}{2}}$.:

Solution

Comparing the equation with (1),

$$\alpha=\tfrac{1}{4},\ p(0)=0,\ \lambda=1,\ K(t,s)=ts-t^2s^2,\ f(t)=\frac{3\sqrt{\pi}}{4\Gamma\left(\frac{13}{6}\right)}t^{\frac{7}{6}}-\frac{2}{63}t^{\frac{9}{2}}\left(9-7t^2\right)$$

Writing the equation in the integral form, we have

$$u(t)=\frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^t (t-s)^{-\frac{3}{4}}\left[\frac{3\sqrt{\pi}}{4\Gamma\left(\frac{13}{6}\right)}t^{\frac{7}{6}}-\frac{2}{63}t^{\frac{9}{2}}\left(9-7t^2\right)\right]ds$$
$$+\frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^t (t-s)\left[\int_0^s \left(\tau s-\tau^2 s^2\right)u(\tau)d\tau\right]ds$$

Substituting the value of the integrals, we have

$$u(t)=\frac{3\sqrt{\pi}}{4\Gamma\left(\frac{29}{12}\right)}t^{\frac{17}{12}}+\frac{2\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{13}{2}\right)}t^{\frac{11}{2}}+\frac{2\Gamma\left(\frac{15}{2}\right)}{9\Gamma\left(\frac{31}{4}\right)}t^{\frac{27}{4}}$$
$$+\frac{1}{\Gamma\left(\frac{1}{4}\right)}\left[\frac{\Gamma(i+4)}{\Gamma\left(\frac{17}{4}+i\right)}t^{\left(\frac{13}{4}+i\right)}\right]\beta A+\frac{1}{\Gamma\left(\frac{1}{4}\right)}\left[\frac{\Gamma(i+6)}{\Gamma\left(\frac{25}{4}+i\right)}t^{\left(\frac{21}{4}+i\right)}\right]\beta A$$

Taking N = 2 for illustration,

$$X(t)=\begin{bmatrix}1 & t & t^2\end{bmatrix},\ A=\begin{bmatrix}a_0 & a_1 & a_2\end{bmatrix}^T,\ x_i=\begin{bmatrix}1 & \frac{1}{2} & 1\end{bmatrix}$$

$$u(t_i)=\begin{bmatrix}0 & \frac{2^{(1/2)}}{4}-\frac{5591749427583147 5\times 2^{(1/6)}}{9223372036854775808} & \frac{5185064906597637 5}{72057594037927936}\end{bmatrix}$$

$$A=\begin{bmatrix}\frac{369}{917} & \frac{357}{715} & \frac{491}{5067}\end{bmatrix}$$

$$u_2(t)=0.5814090999\ 47617\,t^2+0.4171918366\ 64559\,t+1.3877787807\ 8145\,e-17$$

$$u_3(t)=-0.248411513995132t^3+0.965590440678948t^2+0.283176056912131t$$
$$+3.12250225675825e-17$$

$$u_5(t)=-0.330182847225224t^5+1.09318495473105t^4-1.52032557010102t^3$$
$$+1.57192671287835t^2+0.18543074911174t-4.33680868994202e-18$$

Table 5: Result for Example 3.

t	N = 3	N = 5	N = 8	Exact Solution
0.2	0.093271537	0.089444051	0.089413645	0.08944272
0.4	0.251866560	0.252984200	0.252979410	0.25298221
0.6	0.463861310	0.467634900	0.464760880	0.46475800
0.8	0.717332030	0.715545250	0.715552020	0.71554175
1.0	1.000355000	1.000034000	0.999999090	1.00000000

t	$ERR_{N=3}$	$ERR_{N=5}$	$ERR_{N=8}$
0.2	3.8288e-03	1.3321e-06	2.9074e-05
0.4	1.1116e-03	1.9869e-06	2.8065e-06
0.6	8.9670e-04	5.4934e-06	2.8761e-06
0.8	1.7903e-03	3.4932e-06	2.6300e-05
1.0	3.5498e-04	3.4000e-05	9.0990e-07

6. Conclusion

This research has developed a shifted Chebyshev collocation method for the numerical solution of initial value problems of fractional order. The shifted Chebyshev polynomial as a basis function provided us with distinct collocation points from zeros to N's. The method made conversion of integro-differential equations easy and also the evaluation of integrals simple. The performance of the shifted Chebyshev collocation method is improved with the increase in the degree of the polynomial as a result of the increase in the step length. Results from the numerical solutions of the fractional order linear and nonlinear integro-differential equations of Volterra type show that the method is computationally reliable.

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