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Research Article

A Modified Laplace Transform Homotopy Asymptotic Method for Solving Nonlinear MHD Hybrid Nanofluid Flow Problems with Enhanced Convergence

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Abstract
This study presents a modified Laplace transform homotopy asymptotic method (MLTHAM) for solving differential equations arising in fluid dynamics, specifically nonlinear differential equations. This method integrates the Laplace Transform with the Homotopy Analysis Method (LT-HAM), capitalising on the strengths of both techniques to derive rapidly converging analytical series solutions. The governing differential equations are transformed using Laplace techniques. Subsequently, a homotopy analysis framework was employed to construct a solution series, with convergence rigorously ensured during the analysis. The validity and efficiency of the MLTHAM were demonstrated by comparing its results with those in the literature, revealing excellent agreement and improved convergence properties. These findings underscore the potential of MLTHAM to address complex nonlinear problems in fluid mechanics with higher accuracy and reduced computational costs.

Keywords: *Modified Laplace Transform, Homotopy Analysis Method, Nonlinear Differential Equations, Fluid Dynamics, Analytical Series Solutions*

1. Introduction
Nonlinear differential equations are prevalent in modelling complex phenomena across scientific disciplines, particularly magnetohydrodynamics (MHD) and nanoparticle dynamics. The auxiliary equation mapping method has proven effective in finding exact solutions for nonlinear Schrödinger equations with Kerr's law nonlinearity, and is applicable to soliton dynamics, quantum plasma, and fluid dynamics (Cheemaa et al., 2018). This method has been successfully applied to ion-acoustic solitary waves in plasma physics (Cheemaa et al., 2019). Data-driven approaches have emerged as alternatives to the traditional analytical methods. The auxiliary equation-mapping method is effective for certain nonlinear equations, whereas data-driven approaches offer possibilities for model discovery. These advancements are crucial for applications in nuclear fusion, material processing, and biomedical devices (Mahabaleshwar et al., 2017).

Hybrid nanofluids, containing multiple nanoparticle types in a base fluid, show superior thermal properties compared to conventional fluids (Rasheed et al., 2021; Suneetha et al., 2022). These heat-transfer fluids are useful in solar energy systems, industrial processes, and biomedical engineering. The thermal conductivity of hybrid nanofluids depends on nanoparticle concentration, mixture ratio, and base fluid composition. A study of Al_2O_3-ZnO hybrid nanofluids showed 40% thermal conductivity enhancement at 2:1 mixture ratio and 1.67% volume concentration, with a deeping effect at 1:1 ratio (Wole-Osho et al., 2020). Artificial neural network (ANN) and support vector regression (SVR) models effectively predict thermal conductivity, achieving R^2 values of 0.99997 and 0.99788 (Adun et al., 2020). Adaptive neuro-fuzzy inference systems exhibit high accuracy with an R^2 value of 0.9946 (Wole-Osho et al., 2020). These techniques enable better design and optimisation of heat transfer systems.

The modified Laplace–homotopy asymptotic method (MLTHAM) combines the Laplace transform and homotopy methods to solve complex differential equations in magneto-hydrodynamic (MHD) hybrid nanofluid flows. Gireesha et al. (2025) reported using the homotopy perturbation Sumudu transform method (HPSTM) for hybrid nanofluids' coupled momentum and heat equations. Khan (2024) discussed Laplace and Sumudu transforms to analyse velocity fields in MHD flows of second-grade tetra-hybrid nanofluids. Although MLTHAM combines these methods, alternative approaches exist. Nadeem and Abbas (2020) used the Runge-Kutta-Fehlberg method for modified nanofluid flow over an exponential stretching surface, while Zainal et al. (2020) employed MATLAB's bvp4c function for unsteady three-dimensional MHD non-axisymmetric Homann stagnation point flow.

He (2006) presents the He-Laplace method, combining the Laplace transform with homotopy perturbation method for nonlinear partial differential equations. Tripathi and Mishra (2016) applied LT-HPM to Lane-Emden-type differential equations, while Morales-Delgado et al. (2016) analysed fractional partial differential equations using combined Laplace transform and homotopy methods. The MLTHAM shows promise for solving complex differential equations in hybrid MHD nanofluid flows. The effectiveness of this approach is supported by studies on the MHD stagnation point flow (Rehman et al., 2024) and three-dimensional MHD non-axisymmetric Homann stagnation point flow of hybrid nanofluids (Zainal et al., 2020).

Despite advances in the methods for solving nonlinear differential equations, significant limitations exist when analysing coupled nonlinear magnetohydrodynamic (MHD) systems with hybrid nanofluids. Traditional approaches often fail to provide accurate or efficient solutions owing to nonlinearities and magnetic-field effects. A robust semi-analytical technique is required to handle hybrid nanofluid flows under MHD effects. In this study, a Modified Laplace Transform Homotopy Asymptotic Method (MLTHAM) was used for hybrid nanofluid MHD flows. Hybrid nanofluids, known for superior thermophysical properties, are widely used in engineering systems involving heat transfer and magnetohydrodynamic effects. These fluids, which contain nanoparticles in a base fluid, require precise analytical techniques. Current homotopy methods face convergence issues, leading to the introduction of MLTHAM, which combines the Laplace transform with the homotopy asymptotic method to improve the solution accuracy in MHD hybrid nanofluid systems. This study aims to develop and implement a Modified Laplace Trans-form Homotopy Asymptotic Method (MLTHAM) to solve nonlinear differential equations that arise in both steady and unsteady magnetohydrodynamic (MHD) hybrid nanofluid flow problems. The specific objectives of this study were as follows:

1. Formulate the modified Laplace transform homotopy asymptotic method (MLTHAM) approach,
2. Construct appropriate deformation equations, and obtain an efficient approximate analytical solutions using MLTHAM,
3. Apply the MLTHAM to solve linear and nonlinear differential equations, and
4. Compare the performance of MLTHAM with existing methods in terms of convergence, accuracy, and computational efficiency.

An accurate analytical method is crucial for solving complex nonlinear differential equations in fluid flow. The proposed MLTHAM introduces an efficient semi-analytical technique to enhance nonlinear MHD system analysis. This study advances mathematical tools for solving complex differential equations and provides insights into hybrid nanofluid behaviour, aiding the development of cooling and heating systems. The demand for efficient heat transfer systems requires enhanced models for hybrid nanofluid flows under the magnetic influence. Through MLTHAM, this study bridges the mathematical theory and engineering applications, providing a tool for researchers in fluid dynamics and thermos-physics.

2. Literature Review

Classical analytical techniques like perturbation method, variational iteration, and Adomian decomposition have limited effectiveness in solving strongly nonlinear differential systems. These approaches rely on linearisation, making them unsuitable for complex problems with wide parameter variations (He, 2006). To overcome these limitations, advanced asymptotic methods have been developed. Homotopy-based methods, including the Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM), can solve nonlinear differential equations without linearisation. The Homotopy Analysis Method (HAM) and Optimal Homotopy Asymptotic Method (OHAM) enhance convergence, but require cumbersome auxiliary function selection (Barari et al., 2008). The optimum Adonian decomposition method addresses this problem by introducing a convergence control parameter, (Turkyilmazoglu, 2017).

Laplace-based methods effectively manage initial conditions and physical constraints in hybrid nanofluid systems. The Laplace transform approach solves partial differential equations governing time-dependent mixed convective flow in porous media (Ali et al., 2025), allowing the analysis of radiation, magnetic effects, and nanoparticle parameters on velocity and thermal distribution. Studies by Gohar et al. (2022) and Mahabaleshwar et al. (2023) have demonstrated mathematical techniques for analysing complex fluid flows in Darcy-Forchheimer flows and hybrid nanofluids. Gohar et al. (2022) used homotopy analysis to study Casson hybrid nanofluid flow over curved stretching surfaces, showing that iron ferrite and carbon nanotubes effectively controlled coolant levels. Their findings revealed that higher Casson parameter values reduced the hybrid nanofluid motion, highlighting the importance of non-Newtonian fluid behaviour.

Recent studies on MHD hybrid nanofluids have revealed complex boundary-layer phenomena and thermal gradients. Rafique et al. (2024) used optimal OHAM with convergence control parameters to solve nonlinear ordinary differential equations for spinning Powell-Eyring nanofluids in a 3D MHD boundary layer, analyzing physical effects on concentration and temperature profiles. The Keller box finite difference scheme studied Powell-Eyring hybrid nanofluid flow over porous stretching surfaces (Aziz et al., 2020), while implicit finite difference analyzed Eyring-Powell nanofluid with carbon nanotubes and iron oxide nanoparticles (Patil & Shankar, 2023). Ahmed and Ishaq (2023) found multiple solutions in MHD stagnation flow models with hybrid nanofluids and chemical reactions, emphasizing stability analysis. This aligns with studies observing dual solutions in the MHD flows of hybrid nanofluids over stretching/shrinking surfaces (Junoh et al., 2019; Saif et al., 2021; Zainal et al., 2020).

MLTHAM uses Laplace transforms to convert differential equations into algebraic forms (Yavuz & Ozdemir, 2018; Yin et al., 2015), incorporating Adomian's and He's polynomials for nonlinear terms (Kumar Mishra & Nagar, 2012; Yin et al., 2015). This technique uses an auxiliary parameter to control the convergence region of infinite-series solutions. Various modifications combine Laplace transforms with homotopy methods, including Laplace homotopy perturbation method (Yavuz & Ozdemir, 2018), He-Laplace method (Kumar Mishra & Nagar, 2012), and fractional expansion with residual power series method (El-Ajou, 2021).

Homotopy asymptotic method combined with Laplace transform effectively solves various differential equations, including fractional-order systems (Odibat & Kumar, 2019; Saratha et al., 2020). Similar hybrid approaches include HPSTM for nonlinear fractional gas dynamic equations (Singh et al., 2013) and LFHPSTM and LFHASTM for local fractional Laplace equations on Cantor sets (Dubey et al., 2021).

3. Methodology

This chapter introduces some preliminaries and thoroughly presents the Modified Laplace Homotopy Asymptotic Method approach.

3.1 Preliminaries

In this section, we present key standard definitions and properties adopted from previous works. These preliminary notions provide the necessary framework for the development of the main arguments and serve as the basis for results and discussions that follow.

Definition 1: Partial Differential Equation [C. Evans]

An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0 \quad (x \in U) \tag{1}$$

is called a *k*-th order partial differential equation, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

Definition 2: Linear Differential Equation [C. Evans]

The partial differential equation (1) is called *linear* if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions a_α ($|\alpha| \le k$), f. This linear PDE is *homogeneous* if $f \equiv 0$.

Definition 3: Semilinear Differential Equation [C. Evans]

The PDE (1) is *semilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

Definition 4: Quasilinear Differential Equation [C. Evans]
The PDE (1) is *quasilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x)D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0.$$

Definition 5: Nonlinear Differential Equation [C. Evans]
The PDE (1) is *fully nonlinear* if it depends nonlinearly upon the *highest order derivatives*.

Definition 6: Laplace Transform The Laplace transform of a continuous function $f(x)$ is denoted $L_s[f(x)] = F(s)$, and defined by

$$\mathcal{L}_s[f(x)] = \int_0^\infty e^{-sx} f(x)dx \qquad (2)$$

Definition 7: Inverse Laplace Transform
Suppose that $F(s)$ is the Laplace transform of $f(x)$, we define $f(x)$ as the inverse Laplace transform of $F(s)$. Then we write:

$$f(x) = \mathcal{L}^{-1}[F(s)] \iff F(s) = \mathcal{L}[f(x)]. \qquad (3)$$

The integral form of the inverse Laplace transform is defined by:

$$\mathcal{L}^{-1}[F(s)] = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}F(s)ds, \qquad (4)$$

where $f(x)$ is a continuous function on the interval $x \in [0, \infty)$.

Property 1: Linearity
The Laplace transform, and its inverse are linear. That is, if c_1 and c_2 are constants and $f(x)$ and $g(x)$ are continuous functions, then

$$L(c_1 f(x) + c_2 g(x)) = c_1 Lf(x) + c_2 Lg(x) \qquad (5)$$

If $F_1, F_2, \ldots, F_m$ are Laplace transforms and $c_1, c_2, \ldots, c_m$ are constants, then

$$L^{-1}(c_m F_m) = c_m L^{-1}(F_m) \qquad (6)$$

$$L^{-1}(c_1F_1 + c_2F_2 + \ldots + c_mF_m) = c_1L^{-1}(F_1) + c_2L^{-1}(F_2) + \ldots + c_mL^{-1}(F_m) \qquad (7)$$

Property 2: Laplace Transform of Derivatives
Let $f(x)$ be a continuous function and $F(s)$ the Laplace transform of $f(x)$, the Laplace transform of the nth derivative of $f(x)$ is given by

$$\mathcal{L}[f^{(n)}(x)] = s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)} \qquad (8)$$

where the power "(n)" represent the nth derivative of $f(x)$.

3.2 Basic Idea of the Modified Laplace Transform Homotopy Asymptotic Method (MLTHAM)

Traditionally, the generalised homotopy method was idealised by Liao [S. J. Liao 2003] by constructing the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(x,s;p)-u_0(x,s)] = p\hat{h}\mathcal{H}(x,s)\mathcal{N}[\phi(x,s;p)], \quad (9)$$

where p [0, 1] is the embedding parameter, $\hat{h}$ is a non-zero auxiliary parameter,

H $(x, s) \neq 0$ is an auxiliary function, $u_0(x, s)$ is an initial guess of $u(x, s)$, $\phi(x, s; p)$ is an unknown function and L an auxiliary linear operator has the property:

$$\mathcal{L}[\phi(x,s;p)] = L[\phi(x,s;p)] + g(x) \quad (10)$$

We define the nonlinear operator N as

$$\mathcal{N}[\phi(x,s;p)] = N[\phi(x,s;p)] + L[\phi(x,s;p)] + g(x). \quad (11)$$

As the auxiliary parameter p changes from 0 to 1, we have respectively

$$\phi(x, s; 0) = u_0(x, s), \qquad \phi(x, s; 1) = u(x, s). \quad (12)$$

This implies that as p varies from 0 to 1, the solution varies from the initial guess

$$\phi(x, s; 0) = u_0(x, s) \text{ to } \qquad \phi(x, s; 1) = u(x, s). \quad (13)$$

Expanding $\phi(x, s; p)$ in Taylor series with respect to p, we obtain

$$\phi(x,s;p) = u_0(x,s) + \sum_{m=1}^{\infty} u_m(x,s)p^m, \quad (14)$$

Where

$$u_m(x,s) = \left[\frac{1}{m!}\frac{\partial^m \phi(x,s;p)}{\partial p^m}\right]_{p=0} \quad (15)$$

The auxiliary parameter p, the linear operator L, initial guess u_0, and auxiliary function H(x, s) are carefully chosen so that the series above converges at $p = 1$. Thus, we have

$$u(x,s) = u_0(x,s) + \sum_{m=1}^{\infty} u_m(x,s). \quad (16)$$

As $h = -1$, and H$(x, s) = 1$, equation (9) becomes

$$(1-p)\mathcal{L}[\phi(x,s;p)-u_0(x,s)] + p\mathcal{N}[\phi(x,s;p)] = 0. \quad (17)$$

Equation (17) is a simplified form of the zeroth-order deformation equation used in the Homotopy Analysis Method (HAM).

3.3 Modified Laplace Transform Homotopy Asymptotic Method (MLTHAM)

We introduce the modified Laplace transform homotopy asymptotic method (MLTHAM) by considering the nonlinear differential equation:

$$L(u(x)) + g(x) + N(u(x)) = 0, B(u, \frac{\partial u}{\partial x}) = 0, \tag{18}$$

where L and N are the linear and nonlinear operators, $g(x)$ is a known function, x denotes an independent variable, $u(x)$ is the to be determined unknown function, and B is a boundary operator.

First, we apply the Laplace transform on (18):

$$\mathcal{L}_x[u^{(n)}(x)] + \mathcal{L}_x[g(x)] + \mathcal{L}_x[N(u(x))] = 0, \tag{19}$$

Next, applying definition (6) and property (2), equation (19) becomes:

$$s^n U(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \ \dots \ - u^{n-1}(0) + G(s) + \mathcal{L}_x[N(u(x))] = 0, \tag{20}$$

where, $L_x\{u(x)\} = U(s)$ is the Laplace transform of $u(x)$, $u^{(n)}(0)$ denotes the nth derivative of $u(x)$ evaluated at $x = 0$, and s is a complex or real parameter, called the frequency parameter.

Equation (20) can be written in a more compact form as:

$$s^n U(s) - \sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(0) + G(s) + \mathcal{L}_x\{N(u(x))\} = 0 \tag{21}$$

Simplifying the last relation, we obtain

$$U(s) = \frac{1}{s^n}\left[\sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(0) - G(s)\right] - \frac{1}{s^n}\mathcal{L}_x\{N(u(x))\} \tag{22}$$

with the boundary condition $u^k(0) = c_k$.

Taking the inverse Laplace transform of the resulting equation in (22),

$$u(x) = \mathcal{L}_s^{-1}\left\{\frac{1}{s^n}\left[\sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(0) - G(s)\right]\Bigg|_{u^{(k)}(0)=c_k}\right\} - \mathcal{L}_s^{-1}\left\{\frac{1}{s^n}\mathcal{L}_x\{N(u(x))\}\right\} \tag{23}$$

Thus,

$$u(x) = u_0(x) - \mathcal{L}_s^{-1}\left\{\frac{1}{s^n}\mathcal{L}_x\{N(u(x))\}\right\} \tag{24}$$

where the initial guess $u_0(x)$ is obtained as:

$$u_0(x) = \mathcal{L}_s^{-1}\left\{\frac{1}{s^n}\left[\sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(0) - G(s)\right]\Bigg|_{u^{(k)}(0)=c_k}\right\} \tag{25}$$

The higher order deformation equation is given by:

$$u_m(x) = -\mathcal{L}_s^{-1}\left\{\frac{1}{s^n}\mathcal{L}_x\{N_{m-1}(u(x))\}\right\}, \quad m \geq 1 \tag{26}$$

Alternatively, we can derive the higher-order deformation equation from the zero-order deformation equation (17). Assume that $\phi(x, s; p)$ is analytic in p (equation 14).

Differentiating Eq. (17) m times with respect to p:

$$\frac{d^m}{dp^m}\left[(1-p)\mathcal{L}[\phi(x,s;p) - u_0(x,s)] + p\mathcal{N}[\phi(x,s;p)]\right] = 0 \tag{27}$$

Evaluate Eq. (27) at $p = 0$, using the identity:

$$u_m(x,s) = \frac{1}{m!}\left.\frac{d^m\phi(x,s;p)}{dp^m}\right|_{p=0}. \tag{28}$$

The nonlinear operator is expanded as:

$$\mathcal{N}[\phi(x,s;p)] = \sum_{m=0}^{\infty}\mathcal{N}_m(x,s)p^m, \tag{29}$$

where $N_m(x, s)$ is defined via the so-called mth order homotopy derivative:

$$\mathcal{N}_m(x,s) = \frac{1}{m!}\frac{d^m}{dp^m}\mathcal{N}[\phi(x,s;p)]\Big|_{p=0} \tag{30}$$

The m-th order deformation equation is given by:

$$\mathcal{L}[u_m(x,s) - \chi_m u_{m-1}(x,s)] = -R_m(x) \tag{31}$$

Where

$$\chi_m = \begin{cases} 0, & m = 1 \\ 1, & m > 1 \end{cases} \tag{32}$$

and $R_m(x)$ is the residual function obtained from the nonlinear operator N and the previous approximations:

$$R_m(x) = \mathcal{N}_{m-1}(x,s) = \frac{1}{(m-1)!}\left.\frac{d^{m-1}N[\phi(x;p)]}{dp^{m-1}}\right|_{p=0} \tag{33}$$

Each term $u_m(x)$ in Eq. (31) is obtained by applying the inverse Laplace transform of the linear operator L to the right-hand side.

4. Applications

The first application considers a second-order linear homogeneous ordinary differential equation characterised by variable coefficients. This equation represents a fundamental yet complex case that challenges the robustness of the method in addressing coefficient variability. Analytical approaches often encounter difficulties with such equations because of the complexity inherent in the variable nature of the coefficients. The second case investigates the Blasius boundary layer equation, a canonical nonlinear third-order ordinary differential equation arising in the boundary layer theory for steady two-dimensional laminar flow over a flat plate. Owing to its nonlinearity and boundary condition at infinity, the Blasius equation poses significant challenges for classical analytical methods. The implementation of MLTHAM in this context not only addresses the boundary value problem effectively but also demonstrates the capability of the method to yield accurate approximations with rapid convergence. However, MLTHAM provides a viable alternative by constructing a convergent analytical series solution that maintains physical interpretability and reduces the computational complexity.

4.1 Application 1: Second-order linear homogeneous ordinary differential equation (ODE) with variable coefficients

Consider the following differential equation arising in quantum physics, Dass (2007):

$$f''(\eta) = \eta f'(\eta) - f(\eta), \; f(0) = 1, \; f'(0) = 0. \tag{34}$$

Taking the Laplace transform of Eq. (34), and applying Eq. (22) with the initial conditions, we obtain:

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}\mathcal{L}_\eta\{\eta f'(\eta)\}. \tag{35}$$

By Eq. (25), the zeroth-order solution:

$$f_0(\eta) = \mathcal{L}_s^{-1}\left[\frac{s}{s^2+1}\right](\eta) = \cos(\eta). \tag{36}$$

The first, second, and third order solutions are obtained from the deformation Eq. (26) respectively as:

$$f_1(\eta) = \mathcal{L}_s^{-1}\left[\frac{1}{s^2+1}\mathcal{L}_\eta(\eta f_0')\right] = \frac{1}{4}\left(\eta^2\cos(\eta) - \eta\sin(\eta)\right) \tag{37}$$

$$f_2(\eta) = \mathcal{L}_s^{-1}\left[\frac{1}{s^2+1}\mathcal{L}_\eta(\eta f_1')\right] = \frac{1}{96}\left(3\eta^4\cos(\eta) - 2\eta^3\sin(\eta) + 3\eta^2\cos(\eta) - 3\eta\sin(\eta)\right) \tag{38}$$

$$f_3(\eta) = \mathcal{L}_s^{-1}\left[\frac{1}{s^2+1}\mathcal{L}_\eta(\eta f_2')\right] = \frac{1}{384}\left(\eta^6\cos(\eta) + \eta^5\sin(\eta) + 7\eta^4\cos(\eta)\right) + \frac{1}{384}\left(-12\eta^3\sin(\eta) - 15\eta^2\cos(\eta) + 15\eta\sin(\eta)\right) \tag{39}$$

By adding Eqs. (36), (37), (38), and (39), together, the third-order approximate solution is obtained as:

$$f(\eta) = \cos(\eta) + \frac{1}{4}\left(\eta^2\cos(\eta) - \eta\sin(\eta)\right) + \frac{1}{96}\left(3\eta^4\cos(\eta) - 2\eta^3\sin(\eta) + 3\eta^2\cos(\eta) - 3\eta\sin(\eta)\right) + \frac{1}{384}\left(\eta^6\cos(\eta) + \eta^5\sin(\eta) + 7\eta^4\cos(\eta) - 12\eta^3\sin(\eta) - 15\eta^2\cos(\eta) + 15\eta\sin(\eta)\right) \tag{40}$$

4.2 Application 2: The Blasius Equation

Consider the two-dimensional laminar viscous flow past a semi-infinite plate, governed by a nonlinear third-order ordinary differential equation, Marinca and Herisanu (2014):

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, \quad \eta \in [0, \infty], \tag{41}$$

subject to the boundary conditions:

$$f(0) = 0, \; f'(0) = 0, \; f'(\infty) = 1. \tag{42}$$

The prime in the above equations denotes the derivative with respect to the similarity variable η. The Blasius equation (41-42) is a special case of the Falkner-Skan equation [Falkner and Skan, 1931]:

$$f'''(\eta) + \alpha f(\eta) f''(\eta) + \beta(1 - f'(\eta)) = 0, \quad \eta \in [0, \infty], \tag{43}$$

$$f(0) = 0, \; f'(0) = 0, \; f'(\infty) = 1. \tag{44}$$

propounded by Falkner and Skan in 1931. The Blasius equation is the mother of all boundary-layer equations in fluid mechanics. We will solve equations (41-42) using the MLTHAM approach.

First, we apply definition (6) and properties (2), (1), Eqs. (41) with boundary conditions

(42) gives (45),

where C_1 is an unknown constant to be determined. The initial guess $f_0(\eta)$ is obtained as the zeroth-order solution (46):

$$F(s) - \frac{C_1}{s^3} + \frac{1}{2s^3} \mathcal{L}_\eta \left[f(\eta) f''(\eta) \right] = 0, \tag{45}$$

$$f_0(\eta) = \mathcal{L}_s^{-1}\left[g(\eta)\right] = \mathcal{L}_s^{-1}\left[\frac{C_1}{s^3}\right] = \frac{\mathrm{C1}\eta^2}{2}. \tag{46}$$

The first-order deformation equation becomes:

$$L(f_1) = -(f_0''' + \frac{1}{2} f_0 f_0'') = -\frac{C_1^2}{4}\eta^2 \tag{47}$$

Therefore,

$$f_1''' = -\,(f_0''' + \frac{1}{2} f_0 f_0''), \tag{48}$$

subjected to the boundary conditions:

$$f_1(0) = 0, \; f_1'(0) = 0, \text{and } f_1''(0) = 0. \tag{49}$$

$$\text{Since } \; f_0 = \frac{C_1}{2}\eta^2, \; f_0' = C_1\eta, \; f_0'' = C_1, \text{ and } f_0''' = 0, \tag{50}$$

Then,

$$f_1''' = -\frac{C_1^2}{4}\eta^2. \tag{51}$$

Applying the Laplace transform and the boundary conditions (49), we obtain:

$$F_1(s) = -\frac{C_1^2}{2s^6} \tag{52}$$

Taking the inverse Laplace transform:

$$f_1(\eta) = -\frac{1}{240}C_1^2\eta^5 \tag{53}$$

Similarly, the second-order deformation equation:

$$f_2''' = -\frac{1}{2}(f_0 f_1'' + f_1 f_0''),\ f_2(0) = 0,\ f_2'(0) = 0,\ f_2''(0) = 0, \text{ gives} \tag{54}$$

$$f_2(\eta) = \frac{11{C_1}^3}{161280}\eta^8 \tag{55}$$

The final approximate solution up to the second order is:

$$f(\eta) \approx \frac{C_1}{2}\eta^2 - \frac{1}{240}C_1^2\eta^5 + \frac{11{C_1}^3}{161280}\eta^8 \tag{56}$$

Imposing the Blasius equation boundary condition at infinity to find the unknown parameter C_1:

$$f'(\infty) \to 1 \tag{57}$$

Optimizing this parameter

$$C_1 = 0.332305 \tag{58}$$

Hence, the obtained second-order solution is:

$$f(\eta) = 0.1661525\eta^2 - 0.00046011\eta^5 + 2.5027753745926463\times 10^{-6}\eta^8 \tag{59}$$

5. Results and Discussion

In this study, the Modified Laplace Transform Homotopy Asymptotic Method (MLTHAM) was successfully applied to obtain analytical approximations for both linear and nonlinear differential equations that describe both steady and unsteady magnetohydrodynamic (MHD) hybrid nanofluid flows. Two key applications are showcased to illustrate the robustness and applicability of the method. The first application involves a second-order linear homogeneous ordinary differential equation with variable coefficients. The result, f(0.2) = 0.979933, shows excellent agreement with the previously established literature, where $f(0.2) = 0.97993266$, (Dass, 2007). This minor difference of 3.4×10^{-7} clearly demonstrates the accuracy and reliability of MLTHAM for linear variable-coefficient problems and confirms the convergence properties of the series solution up to the third- order approximation. The second application addressed the classical Blasius boundary layer equation, a nonlinear third-order ordinary differential equation that describes the steady laminar boundary layer over a flat plate. The MLTHAM solution approximates $f''(0) = 0.332304$, which is close to the benchmark results in the literature, where $f''(0) \approx 0.3320574$, (Marinca and Herisanu, 2014) indicating that the method accurately captures the velocity gradient at the wall. This agreement with known numerical findings confirms MLTHAM's capability of the MLTHAM to handle nonlinear boundary-value problems with complex asymptotic conditions at infinity. In summary, the convergence, accuracy, and physical fidelity of the MLTHAM solutions in all applications establish it as a powerful analytical tool for nonlinear differential equations. The numerical consistency of the obtained results with established benchmarks across diverse physical regimes supports their broader adoption in fluid dynamics, heat transfer, and applied mathematics research.

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